

Interference as Noise: Friend or Foe?

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This version of the paper differs from the one appeared on the IEEE Transactions of Information Theory, Vol. 62, No. 6, pp. 3561-3596: it fixes a problem brought to our attention by Mirza Uzair Baig. The step marked with (d) on page 3585, top left column right above eq.(107), is not correct because the uniform distribution does not provide an upper bound. Because of this, the gap in eq.(108) must contain the additional term $\log(N \sum_{i \in [1:N]} p_i^2)$. This does not affect in any way the results, or the plots, presented in the paper. This is so because the paper uses exclusively equally likely inputs for which the missing term is zero. In what follows, the changes compared to the printed version are highlighted in red.

Abstract

This paper shows that for the two-user Gaussian Interference Channel (G-IC) Treating Interference as Noise without Time Sharing (TINnoTS) achieves the closure of the capacity region to within either a constant gap, or to within a gap of the order $O\left(\log\left(\frac{\ln(\min(S,I))}{\gamma}\right)\right)$ up to a set of Lebesgue measure $\gamma \in [0, 1]$, where S is the largest Signal to Noise Ratio (SNR) on the direct links and I is the largest Interference to Noise Ratio (INR) on the cross links. As a consequence, TINnoTS is optimal from a generalized Degrees of Freedom (gDoF) perspective for all channel gains except for a subset of zero measure. TINnoTS with Gaussian inputs is known to be optimal to within 1/2 bit for a subset of the weak interference regime. Rather surprisingly, this paper shows that TINnoTS is gDoF optimal in *all*

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parameter regimes, even in the strong and very strong interference regimes where joint decoding of Gaussian inputs is optimal.

For approximate optimality of TINnoTS in all parameter regimes it is critical to use non-Gaussian inputs. This work thus proposes to use *mixed inputs* as channel inputs for the G-IC, where a mixed input is the sum of a discrete and a Gaussian random variable. Interestingly, compared to the Han-Kobayashi achievable scheme, the discrete part of a mixed input is shown to effectively behave as a common message in the sense that, although treated as noise, its effect on the achievable rate region is as if it were jointly decoded together with the desired messages at a non-intended receiver. The practical implication is that a discrete interfering input is a “friend,” while an Gaussian interfering input is in general a “foe.” The paper also discusses other practical implications of the proposed TINnoTS scheme with mixed inputs.

Since TINnoTS requires neither explicit joint decoding nor time sharing, the results of this paper are applicable to a variety of *oblivious* or *asynchronous* channels, such as the block asynchronous G-IC (which is not an information stable channel) and the G-IC with partial codebook knowledge at one or more receivers.

Index Terms

Treating Interference As Noise, Interference Channel, Discrete Inputs.

I. INTRODUCTION

Consider the two-user memoryless real-valued additive white Gaussian noise interference channel (G-IC) with input-output relationship

$$Y_1^n = h_{11}X_1^n + h_{12}X_2^n + Z_1^n, \quad (1a)$$

$$Y_2^n = h_{21}X_1^n + h_{22}X_2^n + Z_2^n, \quad (1b)$$

where $X_j^n := (X_{j1}, \dots, X_{jn})$ and $Y_j^n := (Y_{j1}, \dots, Y_{jn})$ are the length- n vector input and output, respectively, for user $j \in [1 : 2]$, the noise vector Z_j^n has i.i.d. zero-mean unit-variance Gaussian components, the input X_j^n is subject to a per-block power constraint $\frac{1}{n} \sum_{i=1}^n X_{ji}^2 \leq 1$, and the channel gains $(h_{11}, h_{12}, h_{21}, h_{22})$ are fixed and known to all nodes. The input X_j^n , $j \in [1 : 2]$, carries the independent message W_j that is uniformly distributed on $[1 : 2^{nR_j}]$, where R_j is the rate and n the block-length. Receiver $j \in [1 : 2]$ wishes to recover W_j from the channel output Y_j^n with arbitrarily small probability of error. Achievable rates and capacity region are defined in the usual way [1]. We shall denote the capacity region by \mathcal{C} .

For simplicity we will focus primarily on the *symmetric* G-IC defined by

$$|h_{11}|^2 = |h_{22}|^2 = S \geq 0, \quad (2a)$$

$$|h_{12}|^2 = |h_{21}|^2 = I \geq 0, \quad (2b)$$

and we will discuss how the results for the symmetric G-IC extend to the general asymmetric setting.

The general *discrete memoryless* IC was introduced in [2] where it was shown that the capacity region of an *information stable IC* [3] is given by

$$\mathcal{C} = \lim_{n \rightarrow \infty} \text{co} \bigcup_{P_{X_1^n X_2^n} = P_{X_1^n} P_{X_2^n}} \left\{ \begin{array}{l} 0 \leq R_1 \leq \frac{1}{n} I(X_1^n; Y_1^n) \\ 0 \leq R_2 \leq \frac{1}{n} I(X_2^n; Y_2^n) \end{array} \right\}, \quad (3)$$

where co denotes the convex closure operation. For the G-IC in (1), the maximization in (3) is further restricted to inputs satisfying the power constraint $\frac{1}{n} \sum_{i=1}^n X_{i,j}^2 \leq 1$, $j \in [1 : 2]$.

An inner bound to the capacity region in (3) can be obtained by considering i.i.d. inputs in (3) thus giving

$$\mathcal{R}_{\text{in}}^{\text{TIN+TS}} = \text{co} \bigcup_{P_{X_1 X_2} = P_{X_1} P_{X_2}} \left\{ \begin{array}{l} 0 \leq R_1 \leq I(X_1; Y_1) \\ 0 \leq R_2 \leq I(X_2; Y_2) \end{array} \right\}, \quad (4)$$

where the superscript “TIN+TS” reminds the reader that the region is achieved by Treating Interference as Noise (TIN)¹ and with Time Sharing (TS), where TS is enabled by the convex hull operation [1]. By further removing the convex hull operation in (4) we arrive at

$$\mathcal{R}_{\text{in}}^{\text{TINnoTS}} = \bigcup_{P_{X_1 X_2} = P_{X_1} P_{X_2}} \left\{ \begin{array}{l} 0 \leq R_1 \leq I(X_1; Y_1) \\ 0 \leq R_2 \leq I(X_2; Y_2) \end{array} \right\}. \quad (5)$$

¹We use the terminology “treating interference as noise” to denote the rates obtained when evaluating expressions for the interference channel of the form

$$\text{Desired rate} \leq I(\text{desired input}; \text{output}),$$

without any other rate expressions, mutual information terms, or explicit rate splits. When evaluated with independent and identically distributed (i.i.d.) Gaussian inputs in the interference channel in (1), these rate expressions look like those in which the interference is indeed treated as noise, i.e.,

$$0 \leq R_i \leq \frac{1}{2} \log \left(1 + \frac{S}{1+I} \right), \quad i \in [1 : 2],$$

where the ‘effective noise’ (at the denominator within the log) looks like the true noise power plus all the interferer’s power. Whether this expression has the same “treating interference as noise” interpretation when using non-Gaussian inputs is open to interpretation, and is one of the focusses of this work. We will however continue to use this terminology.

The region in (5) does not allow the users to time-share. For the G-IC the maximization in (4) and (5) is further restricted to inputs satisfy average power constraint $\frac{1}{n} \sum_{i=1}^n X_{i,j}^2 \leq 1$, $j \in [1 : 2]$.

Obviously

$$\mathcal{R}_{\text{in}}^{\text{TINnoTS}} \subseteq \mathcal{R}_{\text{in}}^{\text{TIN+TS}} \subseteq \mathcal{C}.$$

The question of interest in this paper is how $\mathcal{R}_{\text{in}}^{\text{TINnoTS}}$ fares compared to \mathcal{C} . Note that there are many advantages in using TINnoTS in practice. For example, TINnoTS does not require codeword synchronization, as for example for joint decoding or interference cancellation, and does not require much coordination between users, thereby reducing communications overhead. The goal of this paper is to show that despite its simplicity, TINnoTS approximately achieves the capacity \mathcal{C} .

Next, we review past work relevant to our investigation. We refer the interested reader to [1] for a comprehensive literature survey on general discrete memoryless ICs.

A. Past Work

In general, little is known about the optimizing input distribution in (3) for the G-IC (or in (4) and in (5)) and only some special cases have been solved. In [4] it was shown that i.i.d. Gaussian inputs maximize the sum-capacity in (3) for $\sqrt{\frac{1}{5}}(1+l) \leq \frac{1}{2}$ in the symmetric case. In contrast, the authors of [5] showed that in general multivariate Gaussian inputs do not exhaust regions of the form in (3). The difficulty arises from the competitive nature of the problem [6]: for example, say X_2 is i.i.d. Gaussian, taking X_1 to be Gaussian increases $I(X_1; Y_1)$ but simultaneously decreases $I(X_2; Y_2)$, as Gaussians are known to be the “best inputs” for Gaussian point-to-point power-constrained channels, but are also the “worst noise” (or interference, if it is treated as noise) for a Gaussian input.

Recently in [7], [8], for the G-IC with one oblivious receiver, we showed that a properly chosen discrete input has a somewhat different behavior than a Gaussian input: a discrete X_2 may yield a “good” $I(X_1; Y_1)$ while keeping $I(X_2; Y_2)$ relatively unchanged compared to a Gaussian input, thus substantially improving the rates compared to Gaussian inputs in the same achievable region expression. Moreover, in [7], [8] we showed that treating interference as noise at the oblivious receiver and joint decoding at the other receiver is to within an additive gap of 3.34 bits of the capacity. In this work we seek to *analytically* evaluate the lower bound in (5) for

a special class of *mixed inputs* (a superposition of a Gaussian and a discrete random variable) by generalizing the approach of [7], [8] and show that using TINnoTS at both receivers is to within an additive gap of the capacity. In a way this work follows the philosophy of [9]: the main idea is to use sub-optimal point-to-point codes in which the reduction in achievable rate for the intended receiver is more than compensated by the decrease in the interference created at the other receiver, which results in an overall rate region improvement. In a conference version of this paper [10] we demonstrated that TINnoTS is gDoF optimal and can achieve to within an additive gap the symmetric sum-capacity of the classical G-IC.

Recently there has been lots of interest in characterizing when TIN, with or without TS, is approximately optimal. For example, in [11] “It is shown that in the K -user interference channel, if for each user the desired signal strength is no less than the sum of the strengths of the strongest interference from this user and the strongest interference to this user (all values in dB scale), then the simple scheme of using point to point Gaussian codebooks with appropriate power levels at each transmitter and treating interference as noise at every receiver (in short, TIN scheme) achieves all points in the capacity region to within a constant gap. The generalized degrees of freedom (gDoF) region under this condition is a polyhedron, which is shown to be fully achieved by the same scheme, without the need for time-sharing.” In this paper we aim to show that one can always use TINnoTS and be optimal to within an additive gap in all parameter regimes, and not just in the very weak interference regime identified in [11]. The key is to use more “friendly” codebooks than Gaussian codebooks. We note that for an input constrained additive-noise channel where the noise distribution is arbitrary, Gaussian inputs are known to be optimal to within 0.265 bits [12]; what our work shows is that the same is not true in general in a multiuser competitive scenario.

We are not the first to consider discrete inputs for Gaussian noise channels. Shannon himself pointed out the asymptotic optimality of a Pulse Amplitude Modulation (PAM) input for the point-to-point power-constrained Gaussian noise channel [13, ref.121]. Shannon’s argument was solidified in [14] where firm lower bounds on the achievable rate with a PAM input were derived and used to show their optimality to within 0.41 bits [14, eq.(9)]. In [6], [15] the authors demonstrated the existence of input distributions that outperform i.i.d. Gaussian inputs in $\mathcal{R}_{\text{in}}^{\text{TINnoTS}}$ for certain asynchronous G-IC. Both [6], [15] used local perturbations of an i.i.d. Gaussian input: [15, Lemma 3] considered a fourth order approximation of mutual information, while [6, Theorem 4] used perturbation in the direction of Hermite polynomials of order larger

than three. In both cases the input distribution is assumed to have a density. For the cases reported in [15], [6], the improvement over i.i.d. Gaussian inputs shows in the decimal digits of the achievable rates; it is hence not clear that these classes of inputs can actually provide substantial rate gains compared to Gaussian inputs. In [16], the authors showed that using a discrete input for one user and a Gaussian input for the other user outperforms Gaussian inputs at both users in the TINnoTS region; this gain however was neither shown to be unbounded nor to achieve the capacity to within a gap, as we will do here. Moreover, the approach of [16] was based on bounding the achievable mutual information by using Fano's inequality, similarly to [14, Part a)]; the resulting bounds however will not be tight enough for the purposes of deriving gap results. In this work we generalize the bound due to Ozarow-Wyner in [14, Part b)], which turns out to be sharper than [14, Part a)].

We remark that the optimality of TINnoTS for all channel parameters for the G-IC was pointed out in [1, Remark 6.12]. The proof follows since TINnoTS is always optimal for the Linear Deterministic Approximation (LDA) of the G-IC at high-SNR [17]. Moreover, a scheme for the LDA can be translated into a scheme for the real-valued G-IC that is optimal to within at most 18.6 bits [18, Theorem 2]. This line of reasoning based on a universal gap between the LDA and the G-IC, thus giving a constant gap result, does not provide a concrete practical construction of an approximately optimal scheme. Our proof here extends our original approach in [10] and provides, in closed form, the optimal number of points in the discrete part of the mixed inputs, as well as of the optimal power split among the discrete and continuous parts of the mixed inputs. Moreover, our derived gap is in general smaller than 18.6 bits (this is so because the log-log function grows very slowly in its argument).

We conclude this overview of relevant past work by pointing out that in practice it is well known that a non-Gaussian interference should not be treated as a Gaussian noise. The optimal detector for an additive non-Gaussian noise channel may however be far more complex than a classical minimum-distance decoder. Nonetheless, since the performance increase can be substantial for a moderate complexity increase, Network-Assisted Interference Cancellation and Suppression (NAICS) receivers, which account for the discrete and coded nature of the interference, were adopted in the Long Term Evolution (LTE) Advanced Release 12 [19], [20], [21]. The boost in performance of NAICS-type detectors may be understood as follows. As we pointed out in [8], with TIN the mapping of the codewords to the messages is unknown but the codeword symbols may be known through soft symbol-by-symbol estimation as remarked in [22],

where the authors write “We indeed see that BPSK signaling outperforms Gaussian signaling. This is because demodulation is some form of primitive decoding, which is not possible for the Gaussian signaling.”

B. Contributions and Paper Outline

The main contributions of the paper are as follows:

- 1) In Section II-A, Proposition 1 presents a generalization of a lower bound from [14] on the mutual information attained by a discrete input on a point-to-point additive noise channel and compares its performance with other lower bounds available in the literature.
- 2) In Section II-B, Proposition 2 and Proposition 3 present new bounds on the cardinality and minimum distance of *sum-sets* formed by two discrete constellations. Proposition 4 shows that the set of channel gains for which the cardinality of a sum-set is not equal to the product of the cardinalities of the constituent sets has zero measure.
- 3) Section II-C provides examples of how we intend to use the developed tools. First, we show that discrete inputs are approximately optimal for the point-to-point power-constrained Gaussian channel. Second, we show that a discrete additive state, unknown to both the transmitter and the receiver, degrades performance of a point-to-point power-constrained Gaussian channel by at most a constant gap compared to the case where the state is known at all terminals.
- 4) In Section III, Proposition 5 presents an inner bound obtained by evaluating the TINnoTS region with our proposed mixed inputs, whose performance will then be compared to the outer bound in Proposition 6.
- 5) Section IV focuses on the symmetric G-IC. Theorem 7 shows that TINnoTS with mixed inputs is to within $O(1)$, or $O\left(\log\left(\frac{\ln(\min(S,I))}{\gamma}\right)\right)$ except for a set of Lebesgue measure γ for any $\gamma \in (0, 1]$, of the outer bound in Proposition 6. From this result we infer that:
 - a) The discrete part of the mixed input behaves as a “common message” whose contribution can be removed from the channel output of the non-intended receiver, even though explicit joint decoding of the interference is not employed in TINnoTS;
 - b) The continuous part of the mixed input behaves as a “private message” whose power should be chosen such that it is either received below the noise floor of the non-intended receiver [23], or to have a rate that is approximately half the target rate; and

- c) Time-sharing may be mimicked by varying the number of points in the discrete part of the mixed inputs.
- 6) In Section V we extend the gap result of Theorem 7 to some general asymmetric G-IC's. The channel parameter regime covered in Theorem 8 is such that bounds of the form $2R_1 + R_2$ or $R_1 + 2R_2$ are not active in the outer bound in Proposition 6. The excluded regime, roughly speaking, is such that $\min(|h_{11}|_{\text{dB}}^2, |h_{22}|_{\text{dB}}^2) < |h_{12}|_{\text{dB}}^2 + |h_{21}|_{\text{dB}}^2 < |h_{11}|_{\text{dB}}^2 + |h_{22}|_{\text{dB}}^2$, i.e., the sum of the crosslink gains is upper bounded by the sum of the direct link gains and lower bounded by the minimum of the direct link gains, all quantities expressed in dB scale. Numerical experiments suggest that the insights gained in the symmetric case (see above item 5) hold for the asymmetric case as well and that the proposed TINnoTS with mixed inputs is approximately optimal for the general asymmetric G-IC.
 - 7) In Section VI, Theorem 9 shows that TINnoTS with mixed inputs is gDoF optimal almost everywhere (a.e.), that is, for all channel gains except for an *outage set* of zero measure.
 - 8) In Section VII shows that our approximate optimality results hold for a variety of channels, such as for example the *block-asynchronous* G-IC and the *codebook oblivious* G-IC, thereby demonstrating that lack of codeword synchronism or of codebook knowledge at the receivers results in penalty of at most $O(1)$, or $O\left(\log\left(\frac{\ln(\min(S, I))}{\gamma}\right)\right)$, compared to the classical G-IC.
 - 9) In Section VIII we discuss some practical implications of our TINnoTS with mixed inputs achievability scheme, such as:
 - a) In Section VIII-A we discuss an approximate MAP decoder for the very strong interference regime that is very simple to implement with TINnoTS;
 - b) In Section VIII-B we show through numerical evaluations that our gap results are very conservative and that in practice the achievable rates are much closer to capacity than predicted by our analytical results; and
 - c) In Section VIII-C we show that a gap result can be obtained by using as inputs purely discrete random variables, i.e., to within an additive gap the Gaussian part of the mixed inputs can be replaced by another PAM input.

Section IX concludes the paper. Some proofs can be found in the Appendix.

C. Notation

Throughout the paper we adopt the following notation convention:

- Lower case variables are instances of upper case random variables which take on values in calligraphic alphabets;
- $\log(\cdot)$ denotes logarithms in base 2 and $\ln(\cdot)$ in base e;
- $[n_1 : n_2]$ is the set of integers from n_1 to $n_2 \geq n_1$;
- Y^j is a vector of length j with components (Y_1, \dots, Y_j) ;
- If A is a r.v. we denote its support by $\text{supp}(A)$;
- The symbol $|\cdot|$ may denote different things: $|\mathcal{A}|$ is the cardinality of the set \mathcal{A} , $|X|$ is the cardinality of $\text{supp}(X)$ of the r.v. X , or $|x|$ is the absolute value of the real-valued x ;
- For $x \in \mathbb{R}$ we let $\lfloor x \rfloor$ denote the largest integer not greater than x ;
- For $x \in \mathbb{R}$ we let $[x]^+ := \max(x, 0)$ and $\log^+(x) := [\log(x)]^+$;
- $d_{\min}(\mathcal{S}) := \min_{i \neq j: s_i, s_j \in \mathcal{S}} |s_i - s_j|$ denotes the minimum distance among the points in the set \mathcal{S} . With some abuse of notation we also use $d_{\min(X)}$ to denote $d_{\min(\text{supp}(X))}$ for a r.v. X ;
- Let $f(x), g(x)$ be two real-valued functions. We use the Landau notation $f(x) = O(g(x))$ to mean that for *some* $c > 0$ there exists an x_0 such that $f(x) \leq c g(x)$ for all $x \geq x_0$;
- Operator $\text{co}(\cdot)$ will refer to convex hull operation;
- $X \sim \mathcal{N}(\mu, \sigma^2)$ denotes the density of a real-valued Gaussian r.v. X with mean μ and variance σ^2 ;
- $X \sim \text{PAM}(N, d_{\min(X)})$ denotes the uniform probability mass function over a zero-mean PAM constellation with $|\text{supp}(X)| = N$ points, minimum distance $d_{\min(X)}$, and therefore average energy $\mathbb{E}[X^2] = d_{\min(X)}^2 \frac{N^2-1}{12}$;
- $m(\mathcal{S})$ denotes Lebesgue measure of the set \mathcal{S} ; and
- We let

$$l_g(x) := \frac{1}{2} \log(1 + x), \quad (6)$$

$$N_d(x) := \left\lfloor \sqrt{1 + x} \right\rfloor, \quad (7)$$

$$\begin{aligned} \text{Gap}(x, y) &:= \frac{1}{2} \log \left(x \frac{\pi e}{3} \right) \\ &\quad + \frac{1}{2} \log \left(1 + y \frac{\left(1 + \frac{1}{2} \log(1 + \min(\mathcal{S}, l)) \right)^2}{\gamma^2} \right), \end{aligned} \quad (8)$$

where the subscript d reminds the reader that discrete inputs are involved, while g that

Gaussian inputs are involved. Here $H(X)$ is the entropy of the discrete random variable X , while $h(X)$ is the differential entropy of the absolutely continuous random variable X .

II. MAIN TOOLS

In this Section we present the main tools to evaluate the TINnoTS lower bound in (5) under mixed inputs.

A. Generalized Ozarow-Wyner Bound

At the core of our proofs is the following lower bound on the rate achieved by a discrete input on a point-to-point additive noise channel. The important point here is to derive firm bounds that are valid for *any* discrete constellation at *any* SNR, as opposed to bounds that are either optimized for a fixed SNR, or hold asymptotically in the low or high SNR regimes.

Proposition 1 (Ozarow-Wyner-B bound). *Let X_D be a discrete random variable with minimum distance $d_{\min}(X_D) > 0$. Let Z be a zero-mean unit-variance random variable independent of X_D (not necessarily Gaussian). Then*

$$\begin{aligned} I_d(X_D) &:= [H(X_D) - \text{gap}_{(9)}]^+ \\ &\leq I(X_D; X_D + Z) \leq H(X_D), \end{aligned} \tag{9a}$$

$$\text{gap}_{(9)} := \frac{1}{2} \log \left(\frac{2\pi e}{12} \right) + \frac{1}{2} \log \left(1 + \frac{12}{d_{\min}^2(X_D)} \right). \tag{9b}$$

Proof: The upper bound in (9a) is trivial. The lower bound follows by the approach used in [14, Part b)], where the assumption that X_D is a PAM is not necessary.

For the lower bound, let $\tilde{X} := X_D + U$ with U uniformly distributed on $[-d_{\min}(X_D)/2, +d_{\min}(X_D)/2]$ and independent of X_D and Z , and let $Y := X_D + Z$. Following the approach of [14, eq(15)] via the data processing inequality for $\tilde{X} \rightarrow X_D \rightarrow Y$ we know that

$$I(X_D; Y) \geq I(\tilde{X}; Y) = h(\tilde{X}) - h(\tilde{X}|Y). \tag{10}$$

The assumption that X_D is a PAM used in [14] is not needed and we write [14, eq(16)] as

$$h(\tilde{X}) = H(X_D) + \log(d_{\min}(X_D)). \tag{11}$$

Therefore, it remains to upper bound $h(\tilde{X}|Y)$. The bound follows by the same argument that leads to [14, eq.(19)], which holds under the assumptions of the proposition, i.e., no need to assume a PAM input or a Gaussian noise, and states that for any s^2 and k :

$$h(\tilde{X}|Y) \leq \frac{1}{2} \log(2\pi s^2) + \frac{\log(e)}{2s^2} \mathbb{E}[(\tilde{X} - kY)^2]. \quad (12)$$

Thus, by using $s^2 = \mathbb{E}[(\tilde{X} - kY)^2]$ with $k = \frac{\mathbb{E}[\tilde{X}Y]}{\mathbb{E}[Y^2]}$ we write (12) as

$$h(\tilde{X}|Y) \leq \frac{1}{2} \log \left[2\pi e \left(\frac{d_{\min(X_D)}^2}{12} + \frac{\mathbb{E}[X_D^2]}{\mathbb{E}[X_D^2] + 1} \right) \right]. \quad (13)$$

Combining this, by the non-negativity of mutual information, and since $\frac{\mathbb{E}[X_D^2]}{\mathbb{E}[X_D^2] + 1} \leq 1$, the lower bound in (9a) with the gap expression in (9b) follows immediately. ■

Remark 1. *The proof of Proposition 1 holds for any continuous U such that*

$\text{supp}(U) \subseteq [-d_{\min(X_D)}/2, +d_{\min(X_D)}/2]$. In this case $\log(d_{\min(X_D)})$ must be replaced by $h(U)$ in (11), and $\frac{d_{\min(X_D)}^2}{12}$ must be replaced by the variance of U in (13). However, for this more general case, it may not be easy to analytically express the entropy as a function of the variance, and to relate them to the bound on the size of the support of the distribution given by $d_{\min(X_D)}$.

Remark 2. *In the proof of Proposition 1 we can express $\frac{\mathbb{E}[X_D^2]}{\mathbb{E}[X_D^2] + 1} = \text{lmmse}(X_D|Y)$, that is, the linear minimum mean square error of estimating X_D from observation $Y = X_D + Z$. This term can be tightened and replaced by the minimum mean squared error (MMSE) by using the following relationship between conditional differentiable entropy and MMSE, from [24, Thm. 8.6.6]*

$$h(X|Y) \leq \frac{1}{2} \log((2\pi e) \text{mmse}(X|Y)), \quad (14)$$

and the expression in (13) can be tightened to

$$h(\tilde{X}|Y) \leq \frac{1}{2} \log \left[2\pi e \left(\frac{d_{\min(X_D)}^2}{12} + \text{mmse}(X_D|Y) \right) \right]. \quad (15)$$

The bound in (15) would lead to a smaller gap than in (9b).

Remark 3. *If in Proposition 1 we set $Z = Z_G \sim \mathcal{N}(0, 1)$, then we can tighten the upper bound in (9a) to*

$$I_d(X_D) \leq I(X_D; X_D + Z_G) \leq \min(H(X_D), I_g(\mathbb{E}[X_D^2])), \quad (16)$$

since a Gaussian input is capacity achieving for the power-constrained point-to-point Gaussian noise channel. Moreover, for $|\text{supp}(X_D)| = N$, the mutual information bounds in (9a) are the largest (i.e. maximizes $I_d(X_D)$) for a PAM constellation. This follows from the fact that PAM is uniformly distributed and satisfies with equality the general inequality $H(X_D) \leq \log(N)$.

We next compare the Ozarow-Wyner-B lower bound in Proposition 1 to bounds available in the literature.

Ozarow-Wyner-A, or Fano-based, bound: Proposition 1 generalizes the approach of [14, Part b)]. Had we generalized [14, Part a)], we would have obtained the following lower bound valid for Gaussian noise only

$$[H(X_D) - \text{gap}_{(17)}]^+ \leq I(X_D; X_D + Z_G), \quad (17a)$$

$$\text{gap}_{(17)} := \xi \log \frac{1}{\xi} + (1 - \xi) \log \frac{1}{1 - \xi} + \xi \log(N - 1), \quad (17b)$$

$$\xi := 2Q\left(\frac{d_{\min}(X_D)}{2}\right), \quad (17c)$$

where ξ is the union-of-events upper bound on the probability of symbol error for a minimum-distance symbol-by-symbol detector in Gaussian noise from Fano's inequality. We note that a similar Fano-based bounding technique was also used in [16, Theorem 3].

In the following we are interested in showing that certain upper and lower bounds are to within a constant gap of one another, regardless of the channel parameters. For bounds as in (9), the quantity “gap” upper bounds the difference between the upper and lower bounds. The gap in (17) (that generalizes [14, Part a)] to any discrete input on the Gaussian noise channel) is bounded if the term $\xi \log(N - 1)$ is bounded; by using the Chernoff's bound for the Q-function, i.e., $Q(x) \leq \frac{1}{2}e^{-x^2/2}$ and by imposing $\xi \log(N - 1) \leq 1$, we get

$$\begin{aligned} \text{bounded gap in (17)} &\iff \log(N - 1) \leq e^{d_{\min}^2(X_D)/8} \\ &\iff d_{\min}^2(X_D) \geq 8 \ln(\log(N - 1)), \end{aligned}$$

in other words, the minimum distance squared must be of the order of $\ln(\log(N))$ for the gap in (17) to be bounded. On the other hand, the gap in (9) (that generalizes [14, Part b)] to any discrete input on any additive noise channel) is bounded as long as the minimum distance is lower bounded by a constant; for example

$$\text{bounded gap in (9), say } \text{gap}_{(9b)} \leq \frac{1}{2} \log(6\pi e) \approx 2.047 \text{ bits}$$

$$\Longleftrightarrow d_{\min}(X_D) \geq 2,$$

that is, the minimum distance does not need to grow in a particular way with the number of points of the constellation, but it is required to be bounded by a constant from below.

DTD-ITA'14 bound: In a conference version of this work [7], we derived the following lower bound for the mutual information with a discrete input on a Gaussian noise channel. As before, let the noise $Z_G \sim \mathcal{N}(0, 1)$ be independent of the discrete input X_D , and let $\Pr[X_D = s_j] = p_j > 0$, $j \in [1 : N]$ such that $\sum_{j \in [1:N]} p_j = 1$. We have – the proof can be found in Appendix A:

$$\left[-\log \left(\sum_{(i,j) \in [1:N]^2} \frac{p_i p_j}{\sqrt{4\pi}} e^{-\frac{(s_i - s_j)^2}{4}} \right) - \frac{1}{2} \log(2\pi e) \right]^+ \leq I(X_D; X_D + Z_G), \quad (18a)$$

For uniformly distributed (but not necessarily uniformly spaced) input X_D , the bound in (18a) becomes

$$[\log(N) - \text{gap}_{(18c)}]^+ \leq I(X_D; X_D + Z_G), \quad (18b)$$

$$\text{gap}_{(18c)} := \frac{1}{2} \log\left(\frac{e}{2}\right) + \log\left(1 + (N-1)e^{-d_{\min}^2(X_D)/4}\right). \quad (18c)$$

The advantage of the bound in (18b) (referred to in the following as ‘simple DTD-ITA'14 bound’) is its simplicity: it only depends on the constellation through the number of points and the minimum distance. The bound in (18a) (referred to in the following as ‘full DTD-ITA'14 bound’) is in general tighter than the one in (18b) but requires the knowledge of the whole “distance spectrum” (all pair-wise distances among constellation points) as well as the “shaping” of the constellation (the a priori probability of each constellation point), which does not make it amenable for closed form analytical computations in general.

Again aiming at a bounded gap for a uniformly distributed input, we have

$$\begin{aligned} \text{bounded gap in (18)} &\Longleftrightarrow (N-1)e^{-\frac{d_{\min}^2(X_D)}{4}} \leq 1 \\ &\Longleftrightarrow d_{\min}^2(X_D) \geq 4 \ln(N-1), \end{aligned}$$

in other words, the minimum distance squared must be of the order of $\log(N)$ for the gap in (18c) to be bounded. Because of this ‘strong’ requirement on the minimum distance, in [10] we could show that a mixed input achieves the capacity region of the classical G-IC to within an additive gap of the order of $O\left(\log\left(\frac{\ln(\min(S,I))}{\gamma}\right)\right)$, rather than a constant gap; but it was nonetheless

sufficient to show that TINnoTS with mixed inputs achieves the sum gDoF of the classical G-IC for all channel gains up to a set of zero measure.

Numerical Comparisons: We conclude this subsection by numerically comparing the lower bounds in (9), (17) and (18) for the Gaussian noise channel with a PAM input, which is asymptotically capacity achieving at high SNR [14]. In Fig. 1 we plot bounds on $I(X_D; \sqrt{S} X_D + Z_G)$ vs. S in dB; here S represents the SNR at the receiver, $Z_G \sim \mathcal{N}(0, 1)$ is the noise, and $X_D \sim \text{PAM}\left(N, \sqrt{\frac{12}{N^2-1}}\right)$ is the input with $N = N_d(S) = \lfloor \sqrt{1+S} \rfloor \approx S^{\frac{1}{2}}$. In Fig. 1a we plot the rate bounds while in Fig. 1b the gap to capacity, i.e., the difference between the channel capacity and the different lower bounds. In both figures we show:

- 1) The black curve is the channel capacity $I_g(S)$;
- 2) The blue curve is the Ozarow-Wyner-B bound in (9a). From Fig. 1b this bound is asymptotically (for $S \geq 30\text{dB}$) to within 0.754 bits of capacity, which is much better than the analytic worst case gap of $\frac{1}{2} \log(6\pi e) = 2.8395$ bits shown before;
- 3) The magenta curve is the Ozarow-Wyner-A bound in (17a). This bound is to within $O(\log(S))$ of capacity (i.e., straight line as a function of $S|_{\text{dB}}$);
- 4) The cyan curve is the simple DTD-ITA'14 bound in (18b). Here we used $N = N_d(S^{1-\epsilon}) \approx S^{\frac{1-\epsilon}{2}}$ with $\epsilon = \max\left(0, \frac{\log(\frac{1}{6} \ln(S))}{\log(S)}\right)$. This choice of ϵ was derived in [7, Theorem 3] in order to have a $O(\log \log(S))$ gap to capacity. Had we chosen $\epsilon = 0$ then we could only achieve a ‘gap’ of $O(\log(S))$; Similarly, for the Ozarow-Wyner-A, had we choose the same $\epsilon = \max\left(0, \frac{\log(\frac{1}{6} \ln(S))}{\log(S)}\right)$ a similar $O(\log \log(S))$ gap would have been observed; and
- 5) The green curve is the full DTD-ITA'14 bound in (18a), which from Fig. 1b achieves asymptotically (for $S \geq 30\text{dB}$) to within 0.36 bits of capacity.

The quantity $\frac{1}{2} \log\left(\frac{\pi e}{6}\right)$ is also shown for reference in Fig. 1b; this is the “shaping loss” for a one-dimensional infinite lattice and is the limiting gap if the number of points N grows faster than $S^{1/2}$. The “zig-zag” behavior of the curves at low SNR is due to the floor operation in $N = \lfloor \sqrt{1+S} \rfloor$.

We observe that the relative ranking among the bounds at low SNR (roughly less than 27 dB) is different than at high SNR. In particular we observe a qualitatively different behavior at high SNR: the Ozarow-Wyner-B bound in (9a) (blue curve) and the full DTD-ITA'14 bound in (18a) (green curve) result in a constant gap, while the Ozarow-Wyner-A bound in (17a) (magenta curve) and the simple DTD-ITA'14 bound in (18b) (cyan curve) result in a gap that grows with SNR; this is in agreement with the previous discussion that points out that for a constant gap in

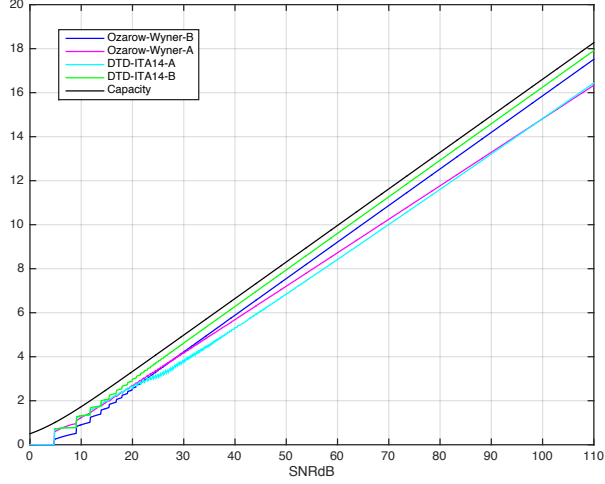
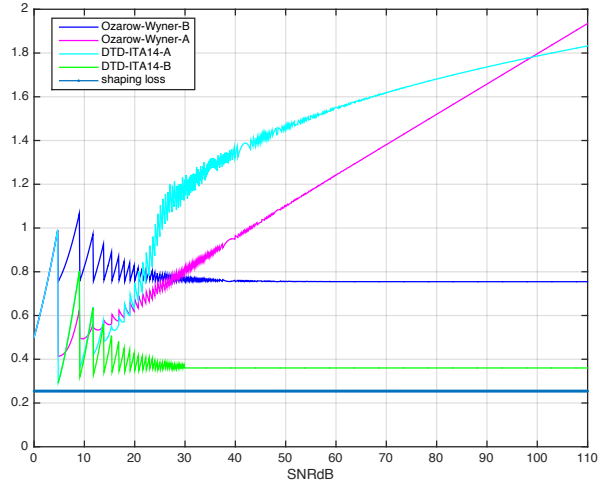
(a) Bounds on $I(X_D; \sqrt{S}X_D + Z_G)$ vs S .(b) Gap to capacity vs S .

Fig. 1: Comparison of different bounds for a PAM input
on a Gaussian noise channel.

the latter two cases the number of points N must grow slower than $S^{1/2}$. The smallest gap at high SNR for $N \cong S^{1/2}$ is given by our full DTD-ITA'14 bound in (18a) (green curve); as pointed out earlier, this bound is unfortunately not amenable for closed form analytical evaluations, so in the following we shall use the Ozarow-Wyner-B bound in (9a) (blue curve) from Proposition 1 whose simplicity comes at the cost of a larger gap.

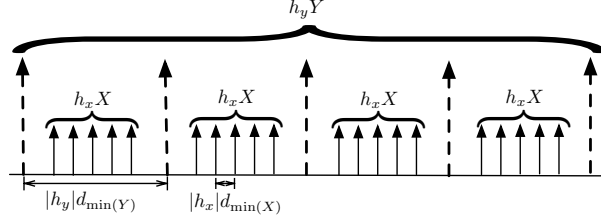


Fig. 2: Structure of the sum-set under the conditions in Proposition 2.

B. Cardinality and Minimum Distance Bounds for Sum-Sets

In multi-user settings, we may wish to select one user's input as Gaussian, another as discrete, or both mixtures of discrete and Gaussian. To handle such scenarios, we need bounds on the cardinality and minimum distance of sums of discrete constellations. If X and Y are two sets, we denote the *sum-set* as

$$X + Y := \{x + y | x \in X, y \in Y\}.$$

Tight bounds on the cardinality and the minimum distance of $X + Y$, for general X and Y , are an open problem in the area of additive combinatorics and number theory [25]. The following set of sufficient conditions for the sum-set obtained with two PAM constellations (actually the probability with which each point is used does not matter as long as it is strictly positive) will play an important role in evaluating our inner bound.

Proposition 2. Let $(h_x, h_y) \in \mathbb{R}^2$ be two constants such that $h_x \cdot h_y \neq 0$. Let $X \sim \text{PAM}(|X|, d_{\min}(X))$ and $Y \sim \text{PAM}(|Y|, d_{\min}(Y))$. Then

$$|h_x X + h_y Y| = |X||Y|, \quad (19)$$

$$d_{\min}(h_x X + h_y Y) = \min(|h_x|d_{\min}(X), |h_y|d_{\min}(Y)), \quad (20)$$

under the following conditions

$$\text{either } |Y||h_y|d_{\min}(Y) \leq |h_x|d_{\min}(X), \quad (21a)$$

$$\text{or } |X||h_x|d_{\min}(X) \leq |h_y|d_{\min}(Y). \quad (21b)$$

Proof: The condition in (21) is such that one PAM constellation is completely contained within two points of the other PAM constellation, see Fig. 2 for a visual illustration. ■

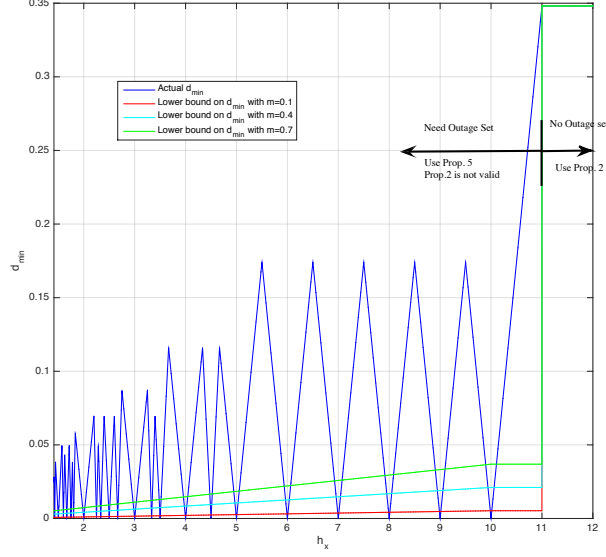


Fig. 3: Minimum distance (blue line) for the sum-set $h_x X + h_y Y$ as a function of h_x for fixed $h_y = 1$ and for $X \sim Y \sim \text{PAM}(10, 1)$. On the right of the vertical green line Proposition 2 is valid. On the left of the vertical green line Proposition 3 must be used; in this case, the minimum distance lower bound in (24a) holds for set of h_x 's for which the blue line is above the red / cyan / green line, where the red, cyan and green lines represent a different value for the measure of the outage set.

We will refer to the condition in (21) as the *non-overlap* condition. Unfortunately, Proposition 2 is not sufficient for our purposes because it restricts the set of channel parameters for which we can compute the minimum distance to those cases where the non-overlap condition holds. When the non-overlap condition in (21) is not satisfied, the minimum distance is very sensitive to the fractional values of h_x and h_y . Fig. 3 shows, in solid blue line, the minimum distance for the sum-set $h_x X + h_y Y$ as a function of h_x for fixed $h_y = 1$ and where X and Y are the same $\text{PAM}(10, 1)$ constellation. It can be observed that there are channel gains for which the minimum distance is zero; those occur on the left of the vertical green line, which separates the values of h_x for which Proposition 2 is valid (right side) for those where it is not (left side).

Remark 4. To bound the cardinality and the minimum distance when the condition in (21) is not satisfied we use the approach of [26], [27]. In [26], [27] it was observed that capacity is sensitive to the fractional values of the channel gains and the channel definition in (1) can be

rewritten in the alternative form as:

$$Y_1^n = h_{11x} \lceil h_{11} \rceil X_1^n + h_{12x} \lceil h_{12} \rceil X_2^n + Z_1^n, \quad (22a)$$

$$Y_2^n = h_{21x} \lceil h_{21} \rceil X_1^n + h_{22x} \lceil h_{22} \rceil X_2^n + Z_2^n, \quad (22b)$$

where h_{ijx} and $\lceil h_{ij} \rceil$ are the fractional and integer parts of the channel gain h_{ij} , respectively. In this representation, the integer part $\lceil h_{ij} \rceil$ captures the magnitude and coarse structure of the channel gain, and the fractional part h_{ijx} is thought to capture the finer structure of the channel gain.

The 'zig-zag' behavior shown on Fig. 3 mainly depend on the fractional part of the channel gains. Following the approach of [26], [27], we define the "outage set"- that is the set of fractional channel gains (for fixed integer part) for which the minimum distance falls below a given target. Moreover, the size of the outage set and the target minimum distance are tunable parameters.

Finally, we remark that for $X \sim \text{PAM}(|X|, d_{\min(X)})$ and $Y \sim \text{PAM}(|Y|, d_{\min(Y)})$ the resulting sum-set given by $h_x X + h_y Y$ can always be restated as

$$h_x X + h_y Y = h_{xx} \lceil h_x \rceil X + h_{yy} \lceil h_y \rceil Y = h_{xx} \hat{X} + h_{yy} \hat{Y}, \quad (22c)$$

where $(h_{xx}, h_{yy}) \in [0, 1]^2$ and where $\hat{X} \sim \text{PAM}(|X|, d_{\min(\lceil h_x \rceil X)})$ and $\hat{Y} \sim \text{PAM}(|Y|, d_{\min(\lceil h_y \rceil Y)})$. Therefore, for the remainder of the paper, we assume that the integer parts $\lceil h_x \rceil, \lceil h_y \rceil$ are fixed and we consider Lebesgue measure over the fractional parts $(h_{xx}, h_{yy}) \in [0, 1]^2$.

We will use the following result to bound the cardinality and the minimum distance when the condition in (21) is not satisfied.

Proposition 3. *Let $X \sim \text{PAM}(|X|, d_{\min(X)})$ and $Y \sim \text{PAM}(|Y|, d_{\min(Y)})$. Then for $(h_{xx}, h_{yy}) \in [0, 1]^2$*

$$|h_{xx} X + h_{yy} Y| = |X||Y| \text{ almost everywhere (a.e.)}, \quad (23)$$

and for any $\gamma > 0$ there exists a set $E \subseteq [0, 1]^2$ such that for all $(h_{xx}, h_{yy}) \in E$

$$d_{\min(h_{xx} X + h_{yy} Y)} \geq \kappa_{\gamma, |X|, |Y|} \cdot \min(|h_{xx}| d_{\min(X)}, |h_{yy}| d_{\min(Y)}, \Upsilon_{|h_{xx}|, |h_{yy}|, |X|, |Y|}), \quad (24a)$$

$$\kappa_{\gamma, |X|, |Y|} := \frac{\gamma/2}{1 + \ln(\max(|X|, |Y|))}, \quad (24b)$$

$$\Upsilon_{|h_{xx}|, |h_{yy}|, |X|, |Y|} := \max \left(\frac{|h_{xx}| d_{\min(X)}}{|Y|}, \frac{|h_{yy}| d_{\min(Y)}}{|X|} \right), \quad (24c)$$

where the Lebesgue measure of the complement of the set E ($E^c = [0, 1]^2 \setminus E$ is referred to as the outage set) satisfies $m(E^c) \leq \gamma$.

Proof. The proof can be found in Appendix B.² □

The reason we need to introduce an outage set in Proposition 3 is that there are values of (h_x, h_y) for which the minimum distance is zero, as it can be seen from Fig. 3. In computing the gap later on, we want to exclude the set of channel gains for which the minimum distance is too close to zero; the measure of this set can be controlled through the parameter γ . The green, cyan, and red lines in Fig. 3 represent lower bounds on the minimum distance that are valid everywhere except for a set of measure no greater than $\gamma = 0.1, 0.3$ and 0.7 , respectively. It is important to notice that the set of channel gains for which the minimum distance is exactly zero satisfies:

Proposition 4. *Let $X \sim \text{PAM}(|X|, d_{\min(X)})$ and $Y \sim \text{PAM}(|Y|, d_{\min(Y)})$. Then the set of $(h_{xx}, h_{yy}) \in [0, 1]^2$ such that $d_{\min(h_{xx}X + h_{yy}Y)} = 0$ has measure zero.*

Proof. The proof follows by observing that the set of channel gains for which $d_{\min(h_{xx}X + h_{yy}Y)} = 0$ and $|h_x X + h_y Y| \neq |X||Y|$ are equivalent and given by eq.(109) in Appendix B. The rest of the proof is similar to that of Proposition 3. □

Remark 5. *Different minimum distance bounds for sum-sets based on Diophantine approximations were used in [28]. For example, consider the sum-set $h_1 X + h_2 X$, i.e., both transmitters use the same PAM constellation X , where $h_1^2 = h_S^2 S$ and $h_2^2 = h_I^2 S^\alpha$ for some fixed $(h_S, h_I) \in \mathbb{R}^2$ and $\alpha > 0$. The authors of [28] focused on the degrees of freedom (DoF) for the case when $\alpha = 1$; in this case the minimum distance can be lower bounded as follows*

$$\begin{aligned} d_{\min(h_1 X + h_2 X)} &= \min_{x_{1i}, x_{2i} \in X} |h_1 x_{1i} - h_2 x_{2i}| \\ &= \min_{z_{1i}, z_{2i} \in [-\frac{N}{2}, \frac{N}{2}]} |h_S \sqrt{S} d_{\min(X)} z_{1i} - h_I \sqrt{S} d_{\min(X)} z_{2i}| \end{aligned}$$

²In our conference paper [10], the minimum distance bound in [10, eq.(8)] was missing the term $\Upsilon_{|h_x|, |h_y|, |X|, |Y|}$ in (24c). However, this did not impact the claimed gDoF results.

$$= \sqrt{S} d_{\min(X)} \min_{z_{1i}, z_{2i} \in [-\frac{N}{2}, \frac{N}{2}]}, |h_S z_{1i} - h_I z_{2i}| \quad (25a)$$

$$\geq \kappa_\epsilon \frac{2^\epsilon}{N^\epsilon} \sqrt{S} d_{\min(X)}, \quad (25b)$$

where the inequality in (25b) comes from Diophantine approximation results, specifically from the Khintchine-Groshev theorem, and says that for almost all real numbers (h_S, h_I) and for any $\epsilon > 0$ there exists a constant $\kappa_\epsilon > 0$, whose analytical expression is not known, such that the bound in (25b) holds.

Unfortunately, bounds such as (25b) are only well suited for the derivation of DoF (i.e. $\alpha = 1$) but not for gDoF (i.e. $\alpha \neq 1$), which is of interest here. The fundamental problem is that for $\alpha \neq 1$, the factorization in (25a) is no longer possible and κ_ϵ may end up being a function of S and α . Moreover, the fact that we have auxiliary constants ϵ and κ_ϵ in (25b), and where κ_ϵ is essentially not known in closed form, makes derivation of closed form gap results very difficult.

C. Examples

In this Section we give an example of how we intend to use discrete inputs in the TINnoTS region in (5) for the G-IC by considering the familiar point-to-point power-constrained additive white Gaussian noise channel. The goal is to derive some properties / results for a simple setting that we shall use often in the subsequent sections. Specifically, we aim to show that the unit-energy discrete input X_D with a properly chosen number of points $N = |\text{supp}(X_D)|$ as a function of S achieves, roughly speaking (\approx)

$$I(X_D; \sqrt{S}X_D + Z_G) \approx \log(N), \quad Z_G \sim \mathcal{N}(0, 1), \quad (26)$$

$$I(X_G; \sqrt{S}X_G + X_D + Z_G) \approx \log(S), \quad X_G \sim \mathcal{N}(0, 1), \quad (27)$$

that is, the discrete input X_D is a “good” input and a “good” interference. To put it more clearly, when we use a discrete constellation as input, as in (26), the mutual information is roughly equal to the entropy of the constellation, which is highly desirable. On the other hand, if the interference, unknown to transmitter and receiver, is from a discrete constellation as in (27), the mutual information is roughly as if there was no interference, which is again highly desirable. In contrast, a Gaussian input instead of X_D would be the “best” input for (26) but the “worst” interference/noise in (27). We next formalize the approximate statements in (26) and (27).

Gaussian Channel: Consider the point-to-point power-constrained Gaussian noise channel

$$Y = \sqrt{S} X + Z_G, \quad (28a)$$

$$\mathbb{E}[X^2] \leq 1, \quad Z_G \sim \mathcal{N}(0, 1), \quad (28b)$$

where X is the information carrying signal, independent of the noise Z_G . The capacity of this channel, as a function of the SNR S , is $C(S) = I_g(S)$ and is achieved by $X \sim \mathcal{N}(0, 1)$ for every S . Consider now the input $X = X_D \sim \text{PAM}\left(N, \sqrt{\frac{12}{N^2-1}}\right)$ on the channel in (28). By Proposition 1 and Remark 3

$$\begin{aligned} & \left[\log(N) - \frac{1}{2} \log\left(\frac{2\pi e}{12}\right) - \frac{1}{2} \log\left(1 + \frac{N^2-1}{S}\right) \right]^+ \\ & \leq I(X_D; \sqrt{S}X_D + Z_G) \leq I_g(S). \end{aligned} \quad (29)$$

By observing the bounds in (30a), we see that for a PAM input to be optimal to within a constant gap we need that $\log(N) \approx I_g(S)$ and that $\frac{N^2-1}{S}$ is upper bounded by a constant. By choosing $N = \lfloor \sqrt{1+S} \rfloor =: N_d(S)$ it is easy to see that a PAM input can achieve the capacity $I_g(S)$ to within $\frac{1}{2} \log\left(\frac{2\pi e}{3}\right) \approx 1.25$ bits, where the maximum gap is for $S = 3 - \epsilon$ for some $0 < \epsilon \ll 1$.

Note that, had we kept the term $\frac{\mathbb{E}[X_D^2]}{\mathbb{E}[X_D^2]+1}$ in (13), the bound in (30a) would have had $\frac{N^2-1}{S+1}$ in place of $\frac{N^2-1}{S}$ and would have resulted in a gap of at most $\frac{1}{2} \log\left(\frac{\pi e}{2}\right) \approx 1.047$ bits. As always, bounds which allow for expressions that are easier to manipulate analytically come at the expense of a larger gap.

Gaussian Channel with States: The above example showed that a discrete input with $\log(N) \approx I_g(S)$ is a “good” input in the sense alluded to by (26). We now show that a discrete interference is a “good” interference in the sense alluded to by (27). We study an extension of the channel in (28) by considering an *additive state* T available neither at the encoder nor at the decoder. The input-output relationship is

$$Y = \sqrt{S} X + h T + Z_G : \quad (30a)$$

$$\mathbb{E}[X^2] \leq 1, \quad Z_G \sim \mathcal{N}(0, 1), \quad (30b)$$

$$T \text{ discrete with finite power.} \quad (30c)$$

It is well known [1, Section 7.4] that the capacity of the channel with random state in (30) is

$$C = \max_{P_X} I(X; Y) \leq \max_{P_X} I(X; Y|T) = I_g(S). \quad (31)$$

From [12] we know that $X = X_G \sim \mathcal{N}(0, 1)$ is at most 1/2 bit from the capacity C , but the value of the capacity is unknown. In particular it is not known whether the gap to the interference free capacity $I_g(S) - C$ is a bounded function of S .

Assume we use the input $X = X_G \sim \mathcal{N}(0, 1)$, as a Gaussian input is not too bad for an additive noise channel [12]; assume also that $d_{\min}(T) > 0$; then the capacity C is lower bounded as:

$$C \geq I_g(S) - \text{gap}_{(32)}, \quad (32a)$$

$$\text{gap}_{(32)} := \frac{1}{2} \log \left(\frac{2\pi e}{12} \right) + \frac{1}{2} \log \left(1 + \frac{12}{d_{\min}^2(T)} \right), \quad (32b)$$

since

$$\begin{aligned} I(X_G; Y) &= I(X_G; \sqrt{S} X_G + h T + Z_G) \\ &= \underbrace{h(\sqrt{S} X_G + h T + Z_G) - h(\sqrt{S} X_G + Z_G)}_{\geq I_d\left(\frac{h}{\sqrt{1+S}}T\right) \geq H(T) - \text{gap}_{(32)}} \\ &\quad - \underbrace{(h(h T + Z_G) - h(Z_G))}_{\leq H(T)} \\ &\quad + \underbrace{(h(\sqrt{S} X_G + Z_G) + h(Z_G))}_{=I_g(S)}. \end{aligned}$$

Thus, as long as $d_{\min}(T)$ is lower-bounded by a constant, it is possible to achieve the interference-free capacity to within the constant gap in (32b) even when the state is unknown to both the transmitter and the receiver.

The expression in (32) can be readily used to obtained inner bounds on the capacity region of a G-IC where one user has a Gaussian input and the other a discrete input and where the discrete input is treated as noise, as we shall do in the next sections.

III. TINNOTS WITH MIXED INPUTS ACHIEVABLE RATE REGION AND AN OUTER BOUND FOR THE G-IC

For the G-IC in (1) we now evaluate the TINnoTS region in (5) with inputs

$$X_i = \sqrt{1 - \delta_i} X_{iD} + \sqrt{\delta_i} X_{iG}, \quad i \in [1 : 2] : \quad (33a)$$

$$X_{iD} \sim \text{PAM} \left(N_i, \sqrt{\frac{12}{N_i^2 - 1}} \right), \quad (33b)$$

$$X_{iG} \sim \mathcal{N}(0, 1), \quad (33c)$$

$$\mathbf{p} := [N_1, N_2, \delta_1, \delta_2] \in \mathbb{N} \times \mathbb{N} \times [0, 1] \times [0, 1], \quad (33d)$$

where the random variables X_{ij} are independent for $i \in [1 : 2]$ and $j \in \{D, G\}$. The input in (33) has four parameters, collected in the vector \mathbf{p} , namely: the number of points $N_i \in \mathbb{N}$ and the power split $\delta_i \in [0, 1]$, for $i \in [1 : 2]$, which must be chosen carefully in order to match a given outer bound.

Proposition 5. *For the G-IC the TINnoTS region in (5) contains the region \mathcal{R}_{in} defined as*

$$\mathcal{R}_{in} := \bigcup \left\{ \begin{array}{l} 0 \leq R_1 \leq \mathsf{I}_d(S_1) + \mathsf{I}_g\left(\frac{|h_{11}|^2 \delta_1}{1 + |h_{12}|^2 \delta_2}\right) \\ \quad - \min\left(\log(N_2), \mathsf{I}_g\left(\frac{|h_{12}|^2 (1 - \delta_2)}{1 + |h_{12}|^2 \delta_2}\right)\right) \\ 0 \leq R_2 \leq \mathsf{I}_d(S_2) + \mathsf{I}_g\left(\frac{|h_{22}|^2 \delta_2}{1 + |h_{21}|^2 \delta_1}\right) \\ \quad - \min\left(\log(N_1), \mathsf{I}_g\left(\frac{|h_{21}|^2 (1 - \delta_1)}{1 + |h_{21}|^2 \delta_1}\right)\right) \end{array} \right\}, \quad (34)$$

where the union is over all possible parameters $[N_1, N_2, \delta_1, \delta_2] \in \mathbb{N}^2 \times [0, 1]^2$ for the mixed inputs in (33) and where the equivalent discrete constellations seen at the receivers are

$$S_1 := \frac{\sqrt{1 - \delta_1} h_{11} X_{1D} + \sqrt{1 - \delta_2} h_{12} X_{2D}}{\sqrt{1 + |h_{11}|^2 \delta_1 + |h_{12}|^2 \delta_2}}, \quad (35a)$$

$$S_2 := \frac{\sqrt{1 - \delta_1} h_{21} X_{1D} + \sqrt{1 - \delta_2} h_{22} X_{2D}}{\sqrt{1 + |h_{21}|^2 \delta_1 + |h_{22}|^2 \delta_2}}. \quad (35b)$$

Proof: Due to the symmetry of the problem we derive a lower bound on $I(X_2; Y_2)$ only by following steps similar to those in (32); a lower bound on $I(X_1; Y_1)$ follows by swapping the role of the users. Let $Z_G \sim \mathcal{N}(0, 1)$. Then:

$$\begin{aligned} I(X_2; Y_2) &= I(X_2; h_{21}X_1 + h_{22}X_2 + Z_G) \\ &= \\ &\underbrace{\left[h \left(\frac{\sqrt{1 - \delta_1} h_{21} X_{1D} + \sqrt{1 - \delta_2} h_{22} X_{2D}}{\sqrt{1 + |h_{21}|^2 \delta_1 + |h_{22}|^2 \delta_2}} + Z_G \right) - h(Z_G) \right]}_{\geq \mathsf{I}_d(S_2) \text{ by Proposition 1}} \\ &\quad - \underbrace{\left[h \left(\frac{\sqrt{1 - \delta_1}}{\sqrt{1 + |h_{21}|^2 \delta_1}} h_{21} X_{1D} + Z_G \right) - h(Z_G) \right]}_{\leq \min\left(\log(N_1), \frac{1}{2} \log\left(1 + \frac{|h_{21}|^2 (1 - \delta_1)}{1 + |h_{21}|^2 \delta_1}\right)\right) \text{ by Remark 3}} \end{aligned}$$

$$+ \frac{1}{2} \log(1 + |h_{21}|^2 \delta_1 + |h_{22}|^2 \delta_2) - \frac{1}{2} \log(1 + |h_{21}|^2 \delta_1).$$

By considering the union over all possible choices of parameters for the mixed inputs we obtain the region in (34), which is contained within the achievable region in (5) and hence forms a lower bound to the capacity region. ■

In the following sections we shall show that our TINnoTS region with mixed inputs in Proposition 5 is to within an additive gap of the outer bound region given by:

Proposition 6. *The capacity region of the G-IC is contained in*

$$\mathcal{R}_{out} = \left\{ \begin{aligned} R_1 &\leq \lg(|h_{11}|^2), \text{ cut-set bound,} \end{aligned} \right. \quad (36a)$$

$$R_2 \leq \lg(|h_{22}|^2), \text{ cut-set bound,} \quad (36b)$$

$$\begin{aligned} R_1 + R_2 &\leq \left[\lg(|h_{11}|^2) - \lg(|h_{21}|^2) \right]^+ \\ &\quad + \lg(|h_{21}|^2 + |h_{22}|^2), \text{ from [29],} \end{aligned} \quad (36c)$$

$$\begin{aligned} R_1 + R_2 &\leq \left[\lg(|h_{22}|^2) - \lg(|h_{12}|^2) \right]^+ \\ &\quad + \lg(|h_{11}|^2 + |h_{12}|^2), \text{ from [29],} \end{aligned} \quad (36d)$$

$$\begin{aligned} R_1 + R_2 &\leq \lg \left(|h_{12}|^2 + \frac{|h_{11}|^2}{1 + |h_{21}|^2} \right) \\ &\quad + \lg \left(|h_{21}|^2 + \frac{|h_{22}|^2}{1 + |h_{12}|^2} \right), \text{ from [23],} \end{aligned} \quad (36e)$$

$$\begin{aligned} 2R_1 + R_2 &\leq \lg(|h_{11}|^2 + |h_{12}|^2) + \lg \left(|h_{21}|^2 + \frac{|h_{22}|^2}{1 + |h_{12}|^2} \right) \\ &\quad + \left[\lg(|h_{11}|^2) - \lg(|h_{21}|^2) \right]^+, \text{ from [23],} \end{aligned} \quad (36f)$$

$$\begin{aligned} R_1 + 2R_2 &\leq \lg(|h_{21}|^2 + |h_{22}|^2) + \lg \left(|h_{12}|^2 + \frac{|h_{11}|^2}{1 + |h_{21}|^2} \right) \\ &\quad + \left[\lg(|h_{22}|^2) - \lg(|h_{12}|^2) \right]^+, \text{ from [23]} \end{aligned} \quad (36g)$$

For the classical G-IC where all nodes are synchronous and possess full codebook knowledge, this outer bound is tight in strong interference $\{|h_{21}|^2 \geq |h_{11}|^2, |h_{12}|^2 \geq |h_{22}|^2\}$ [30] and achievable to within 1/2 bit otherwise [23].

The key step to match, to within an additive gap, the outer bound region \mathcal{R}_{out} in Proposition 6 to our TINnoTS achievable region with mixed inputs \mathcal{R}_{in} in Proposition 5 is to carefully choose

the mixed input parameter vector $[N_1, N_2, \delta_1, \delta_2]$. This ‘carefully picking of the mixed input parameters’ is the objective of Section IV.

IV. SYMMETRIC CAPACITY REGION TO WITHIN A GAP

The main result of this paper is:

Theorem 7. *For the symmetric G-IC, as defined in (2), the TINnoTS achievable region in (34), with the parameters for the mixed inputs chosen as indicated in Table I, and the outer bound in (36) are to within a gap of:*

- *Very Weak Interference:* $S \geq I(1 + I)$:

$$\text{gap} \leq \text{Gap} \left(\frac{6}{\pi e}, 0 \right) = \frac{1}{2} \text{ bits},$$

- *Moderately Weak Interference Type2:* $I \leq S \leq I(1 + I)$, $\frac{1+S}{1+I+\frac{S}{1+I}} > \frac{1+I+\frac{S}{1+I}}{1+\frac{S}{1+I}}$.

$$\text{gap} \leq \text{Gap} \left(\frac{608}{9}, 0 \right) \approx 3.79 \text{ bits},$$

- *Moderately Weak Interference Type1:* $I \leq S \leq I(1 + I)$, $\frac{1+S}{1+I+\frac{S}{1+I}} \leq \frac{1+I+\frac{S}{1+I}}{1+\frac{S}{1+I}}$.

$$\text{gap} \leq \text{Gap}(16, 45),$$

except for a set of measure γ for any $\gamma \in (0, 1]$,

- *Strong Interference:* $S < I < S(1 + S)$:

$$\text{gap} \leq \text{Gap}(2, 8),$$

except for a set of measure γ for any $\gamma \in (0, 1]$,

- *Very Strong Interference:* $I \geq S(1 + S)$:

$$\text{gap} \leq \text{Gap}(2, 0) \approx 1.25 \text{ bits}.$$

Before we move to the proof of Theorem 7, we would like to offer our thoughts on why a $O \left(\log \left(\frac{\ln(\min(S, I))}{\gamma} \right) \right)$ gap is obtained in some regimes up to an outage set of controllable measure (the larger the measure of the channel gains for which the derived gap does not hold, the lower the gap). We start by noticing that, for the symmetric G-IC, whenever the TINnoTS region with our mixed input is optimal to within a *constant gap* then the gap result holds for *all channel gains*. Otherwise, the optimality is to within a $O \left(\log \left(\frac{\ln(\min(S, I))}{\gamma} \right) \right)$ gap and holds for all channel

gains up to an outage set. We found a $O\left(\log\left(\frac{\ln(\min(S,I))}{\gamma}\right)\right)$ gap up to an outage set whenever the sum-rate upper bound $\min(\text{eq.}(36\text{c}), \text{eq.}(36\text{d}))$ is active, which in gDoF corresponds to the regime $\alpha \in (2/3, 2)$ meaning that the interference is neither very weak nor very strong. It is thus natural to ask: (a) whether the $O\left(\log\left(\frac{\ln(\min(S,I))}{\gamma}\right)\right)$ gap and/or the ‘up to an outage set’ condition are necessary (not a consequence of the achievable scheme used), and (b) whether a $O\left(\log\left(\frac{\ln(\min(S,I))}{\gamma}\right)\right)$ gap and the ‘up to an outage set’ condition are necessarily always together. We do not have answers to these questions, but we provide our perspective next.

The sum-rate bounds in (36c) and (36d) were originally derived for the classical two-user IC in Gaussian noise in [29] and then extended to any memoryless two-user IC with source cooperation / generalized feedback in [31], and then to any memoryless cooperative two-user IC (where each node can have an input and an output to the channel) in [32] – see also K -user extensions in [33], [34]. In [32] it was noted that surprisingly these bounds hold for a broad class of two-user IC-type channels, which includes for example cognitive ICs and certain ICs with cooperation. The difference is that the mutual information optimization is over all product input distributions for the classical IC, while it is over all joint input distributions for the cooperative or cognitive IC. The ability to correlate inputs is well known to only increase the rates by a constant number of bits; thus, up to a constant gap, channel models from the basic classical IC to the intricate cognitive IC have the same sum-rate upper bound in some regimes. Note that for the real-valued cognitive G-IC for example, the sum-rate bound is achievable to within 1/2 bit for all channel gains by using Dirty Paper Coding. It is not clear at this point whether the $O\left(\log\left(\frac{\ln(\min(S,I))}{\gamma}\right)\right)$ gap up to an outage set for the classical G-IC is thus a fundamental consequence of the fact that the upper bound can be achieved to within a constant gap with sophisticated coding techniques (such as Dirty Paper Coding for the cognitive G-IC) but not with simpler ones (essentially rate splitting and superposition coding as in the Han-Kobayashi scheme) allowed for the classical G-IC.

Another intriguing observation is that these bounds also determine the optimality of “everybody gets half the cake”-DoF result for the K -user G-IC [35], [28]. For the K -user G-IC with fixed channel gains it is well known that the DoF are discontinuous at rational channel gains [36]. This seems to suggest, at least for $\alpha = 1$, that a gap result up to an outage set is actually *fundamental* and not a consequence of the achievable scheme used. Whether the converse result of [36] for $\alpha = 1$ can be extended to the whole regime $\alpha \in (2/3, 2)$ is an open question. We also note that a constant (not $O\left(\log\left(\frac{\ln(\min(S,I))}{\gamma}\right)\right)$) gap result up to an outage set for the whole

TABLE I: Parameters for the mixed inputs in (33), as used in the proof of Theorem 7.

Notation: for $\mathbf{p} = [N_1, N_2, \delta_1, \delta_2]$ we let $\mathbf{p}' = [N_2, N_1, \delta_2, \delta_1]$. The parameter α measures the level of the interference when $S \rightarrow \infty$ according to the parametrization $I = S^\alpha$.

Regime	Input Parameter \mathbf{p} in (33)	Gap(bits)
$S \geq I(1+I)$, $\alpha \in [0, 1/2]$ (very weak)	$\mathbf{p}_t \cup \mathbf{p}'_t$, for all $t \in [0, 1]$; $\mathbf{p}_t := [1, 1, t, 1]$;	constant gap $\text{gap}_{(76)} \leq 1/2$ bits
$S < I(1+I)$, $\frac{1+S}{1+I+\frac{S}{1+I}} > \frac{1+I+\frac{S}{1+I}}{1+\frac{S}{1+I}}$, $\alpha \in (1/2, 2/3)$ (moderately weak 2)	$\mathbf{p}_{1,t} \cup \mathbf{p}_{2,t} \cup \mathbf{p}'_{2,t}$, for all $t \in [0, 1]$; $\mathbf{p}_{1,t}$: values can be found in (152); $\mathbf{p}_{2,t}$: values can be found in (158);	constant gap $\text{gap}_{(74)} \approx 3.79$ bits
$I \leq S$, $\frac{1+S}{1+I+\frac{S}{1+I}} \leq \frac{1+I+\frac{S}{1+I}}{1+\frac{S}{1+I}}$, $\alpha \in [2/3, 1]$ (moderately weak 1)	$\mathbf{p}_{1,t} \cup \mathbf{p}_{2,t} \cup \mathbf{p}'_{2,t}$, for all $t \in [0, 1]$; $\mathbf{p}_{1,t}$ values can be found in (63); $\mathbf{p}_{2,t}$ values can be found in (125);	log-log gap $\text{gap}_{(67)}$
$S < I < S(1+S)$, $\alpha \in (1, 2)$ (strong)	\mathbf{p}_t , for all $t \in [0, 1]$; \mathbf{p}_t : values can be found in (49);	log-log gap $\text{gap}_{(48)}$
$I \geq S(1+S)$, $\alpha \in [2, \infty)$ (very strong)	$\mathbf{p} = [N_d(S), N_d(S), 0, 0]$;	constant gap $\text{gap}_{(42)} \approx 1.25$ bits

regime $\alpha \in (2/3, 2)$ was found in [27]; in this case the achievable region was based on a *multi-letter* scheme inspired by compute-and-forward. It is not clear at this point whether *single-letter* schemes, such as out TINnoTS, are fundamentally suboptimal compared to multi-letter ones.

Proof: The parameters of the mixed inputs in (33) are chosen as indicated in Table I depending on the regime of operation. We now analyze each regime separately.

A. Very strong interference, i.e., $I \geq S(1+S)$

Outer Bound: In the very strong interference regime the capacity of the classical G-IC is given by

$$\mathcal{R}_{\text{out}}^{(\text{IV-A})} = \left\{ \begin{array}{l} 0 \leq R_1 \leq I_g(S) \\ 0 \leq R_2 \leq I_g(S) \end{array} \right\}. \quad (37)$$

Inner Bound: The capacity of the classical G-IC in this regime is achieved by sending only common messages from Gaussian codebooks; a receiver first decodes the interfering message, strips it from the received signal, and then decodes the intended message in an equivalent

interference-free channel. Even though joint decoding is not allowed in our TINnoTS region, we shall see that the discrete part of the input behaves as a common message (as if it could be decoded at the non-intended destination). We therefore do not send the Gaussian portion of the input (as Gaussian inputs treated as noise increase the noise floor of the receiver) and in (33) we set

$$N_1 = N_2 = N = N_d(S), \quad (38a)$$

$$\delta_1 = \delta_2 = \delta = 0, \quad (38b)$$

resulting in

$$S_1 \sim S_2 \sim S, \quad S := \sqrt{S}X_{1D} + \sqrt{I}X_{2D}, \quad (39)$$

for the received constellations in (35). The number of points and the minimum distance for the constellation S in (39) can be computed from Proposition 2 as follows. If we identify $|h_x|^2 = S$, $|h_y|^2 = I$, $|X| = |Y| = N$, $d_{\min(X)}^2 = d_{\min(Y)}^2 = \frac{12}{N^2-1}$, then the condition in (21) reads $N^2S \leq I$, which is readily verified since $N^2S \leq (1+S)S$ by definition of N in (38a), and $(1+S)S \leq I$ by the definition of the very strong interference regime. We therefore have

$$|S| = N^2, \quad \text{with equally likely points}, \quad (40)$$

$$\frac{d_{\min(S)}^2}{12} = \min\{S, I\} \frac{1}{N^2-1} = \frac{S}{N^2-1}. \quad (41)$$

By plugging these values in Proposition 5, an achievable rate region is

$$\mathcal{R}_{\text{in}}^{(\text{IV-A})} = \left\{ \begin{array}{l} 0 \leq R_1 \leq r_0 \\ 0 \leq R_2 \leq r_0 \end{array} \right\} \text{ such that} \quad (42a)$$

$$\begin{aligned} r_0 &\geq I_d(S) - \min(\log(N), I_g(I)) \\ &\geq \left[\log(N^2) - \frac{1}{2} \log\left(\frac{2\pi e}{12}\right) - \frac{1}{2} \log\left(1 + \frac{N^2-1}{S}\right) \right]^+ \\ &\quad - \log(N) \\ &\geq I_g(S) - \text{gap}_{(42)}, \end{aligned} \quad (42b)$$

$$\text{gap}_{(42)} := \frac{1}{2} \log\left(\frac{2\pi e}{3}\right) = \text{Gap}(2, 0) \approx 1.25 \text{ bits}, \quad (42c)$$

where the gap in (42c) is as for the point-to-point Gaussian channel without states in Section II-C.

Gap: It is immediate to see that the achievable region in (42) and the upper bound in (37) are at most to within $\text{gap}_{(42)}$ bits of one another, where $\text{gap}_{(42)}$ is given in (42c).

B. Strong (but not very strong) interference, i.e., $S < I < S(1 + S)$

Outer Bound: The capacity region of the G-IC in this regime is

$$\mathcal{R}_{\text{out}}^{(\text{IV-B})} = \left\{ \begin{array}{l} 0 \leq R_1 \leq I_g(S) \\ 0 \leq R_2 \leq I_g(S) \\ R_1 + R_2 \leq I_g(S + I) \end{array} \right\} = \bigcup_{t \in [0,1]} \left\{ \begin{array}{l} 0 \leq R_1 \leq \frac{1-t}{2} \log\left(1 + \frac{I}{1+S}\right) + \frac{t}{2} \log(1 + S) \\ \quad =: I_g(S_{0,a,t}) \\ 0 \leq R_2 \leq \frac{1-t}{2} \log(1 + S) + \frac{t}{2} \log\left(1 + \frac{I}{1+S}\right) \\ \quad =: I_g(S_{0,b,t}) \end{array} \right\}, \quad (43)$$

where $t \in [0, 1]$ is the time-sharing parameter (i.e., by varying t we obtain all points on the dominant face of the capacity region described by $R_1 + R_2 = I_g(S + I)$).

Inner Bound: The capacity of the classical G-IC in this regime is achieved by sending only common messages from Gaussian codebooks, and by performing joint decoding of the intended and interfering messages at both receivers. Similarly to the very strong interference regime, we do not send the Gaussian portion of the mixed inputs (i.e., $\delta_1 = \delta_2 = 0$). Differently from the very strong interference regime, here we do not set the number of points of the discrete part of the inputs to be the same for the two users since the corner point of (43) for a fixed t has $R_1 \neq R_2$. Moreover, we lower bound the minimum distance of the sum-set constellations S_1 and S_2 in (35) by using Proposition 3 as follows

$$\frac{d_{\min(S_1)}^2}{12 \kappa_{\gamma, N_1, N_2}^2} \geq \min \left(\frac{S}{N_1^2 - 1}, \frac{I}{N_2^2 - 1}, \max \left(\frac{I}{N_1^2(N_2^2 - 1)}, \frac{S}{N_2^2(N_1^2 - 1)} \right) \right), \quad (44)$$

$$\frac{d_{\min(S_2)}^2}{12 \kappa_{\gamma, N_1, N_2}^2} \geq \min \left(\frac{I}{N_1^2 - 1}, \frac{S}{N_2^2 - 1}, \max \left(\frac{I}{N_1^2(N_2^2 - 1)}, \frac{S}{N_2^2(N_1^2 - 1)} \right) \right), \quad (45)$$

$$\kappa_{\gamma, N_1, N_2} := \frac{\gamma/2}{1 + \ln(\max(N_1, N_2))}, \quad (46)$$

where the minimum distance lower bounds in (44) and (45) hold for all channel gains up to an outage set of Lebesgue measure less than γ for any $\gamma \in (0, 1]$.

By combining the bounds in (44) and (45) we obtain

$$\begin{aligned} & \min_{i \in [1:2]} \frac{d_{\min(S_i)}^2}{12 \kappa_{\gamma, N_1, N_2}^2} \\ & \geq \min \left(\frac{\min(S, I)}{\max(N_1^2, N_2^2) - 1}, \frac{\max(S, I)}{N_1^2 N_2^2 - 1} \right) \\ & \stackrel{\text{for } S \leq I}{=} \min \left(\frac{S}{\max(N_1^2, N_2^2) - 1}, \frac{I}{N_1^2 N_2^2 - 1} \right). \end{aligned} \quad (47)$$

With (47), it can be easily seen that the achievable region in Proposition 5 can be written as the union over all (N_1, N_2) of the region

$$\mathcal{R}_{\text{in}}^{(\text{IV-B})}([N_1, N_2, 0, 0]) = \left\{ \begin{array}{l} 0 \leq R_1 \leq r_1 \\ 0 \leq R_2 \leq r_2 \end{array} \right\} \text{ such that} \quad (48a)$$

$$\begin{aligned} r_1 & \geq I_d(S_1) - \min \left(\log(N_2), I_g(I) \right) \\ & \geq \log(N_1) + \log(2) - \text{gap}_{(48)}, \end{aligned} \quad (48b)$$

$$\begin{aligned} r_2 & \geq I_d(S_2) - \min \left(\log(N_1), I_g(I) \right) \\ & \geq \log(N_2) + \log(2) - \text{gap}_{(48)}, \end{aligned} \quad (48c)$$

$$\begin{aligned} \text{gap}_{(48)} & \leq \log(2) + \frac{1}{2} \log \left(\frac{2\pi e}{12} \right) \\ & \quad + \frac{1}{2} \log \left(1 + \frac{\max \left(\frac{\max(N_1^2, N_2^2) - 1}{S}, \frac{N_1^2 N_2^2 - 1}{I} \right)}{\kappa_{\gamma, N_1, N_2}^2} \right), \end{aligned} \quad (48d)$$

where the expression for $\text{gap}_{(48)}$ comes from the minimum distance expression in (47).

We next need to pick N_1 and N_2 in (48). Our choice is guided by the expression of the ‘compound MAC’ capacity region in this regime given by (43). In our TINnoTS region, time-sharing is not allowed, but varying the number of points of the discrete constellations is; we therefore mimic time-sharing in (43) by choosing as number of points in the discrete part of the mixed inputs as follows: for some fixed t we let

$$N_1 = N_d(S_{0,a,t}), \quad S_{0,a,t} := \left(1 + \frac{I}{1+S} \right)^{1-t} (1+S)^t - 1, \quad (49a)$$

$$N_2 = N_d(S_{0,b,t}), \quad S_{0,b,t} := \left(1 + \frac{I}{1+S} \right)^t (1+S)^{1-t} - 1. \quad (49b)$$

The whole TINnoTS achievable region is obtained by taking union over $t \in [0, 1]$ of the region in (48) with the number of points as in (49).

Gap: Since $\frac{1}{1+S} \leq S \leq I$ by the definition of the strong interference regime, we immediately have that in (49) the equivalent SNRs satisfy $\max(S_{0,a,t}, S_{0,b,t}) \leq S$ for all $t \in [0, 1]$. Thus, for the minimum distance expression in (47), we have

$$\max(N_1^2, N_2^2) - 1 \leq \max(S_{0,a,t}, S_{0,b,t}) \leq S = \min(S, I), \quad (50)$$

$$N_1^2 N_2^2 - 1 \leq (S_{0,a,t} + 1)(S_{0,b,t} + 1) - 1 = S + I \leq 2I. \quad (51)$$

Finally, since $I_g(x) \leq \log(N_d(x)) + \log(2)$, the inner bound in (48) is at most $\text{gap}_{(48)}$ bits from the outer bound in (43), uniformly over all $t \in [0, 1]$, where $\text{gap}_{(48)}$ in (48d) can be further upper bounded thanks to (50)-(51) as

$$\text{gap}_{(48)} \leq \frac{1}{2} \log \left(\frac{2\pi e}{3} \left(1 + \frac{\max(1, 2)}{\kappa_{\gamma, N_1, N_2}^2} \right) \right) \leq \text{Gap}(2, 8), \quad (52)$$

where γ is the Lebesgue measure of the outage set over which the lower bounds on the minimum distance in (47) does not apply. Recall that γ is a tunable parameter that represents a tradeoff between gap and set of channel gains for which the gap result holds, i.e., by increasing the measure of the outage set we can reduce the gap, and vice-versa. A similar behavior was pointed out already in [27].

Remark 6. *Note that, had we been able to use Proposition 2 instead of Proposition 3 to bound the minimum distance of the received constellations, we would have obtained a constant gap result instead of a $O\left(\log\left(\frac{\ln(\min(S, I))}{\gamma}\right)\right)$ gap result. It turns out that in this regime the condition of Proposition 2 is not satisfied – the proof is very tedious and is not reported here for sake of space.*

C. Moderately weak interference, i.e. $I \leq S \leq (1 + I)I$: general setup

The weak interference regime is notoriously more involved to analyze than the other regimes. In this subsection we aim to derive a general framework to deal with the weak (but not very weak) interference regime. Before we move into the gap derivation for this regime, let us summarize the key trick we developed in the strong interference regime to obtain a capacity result to within a gap: write the closure of the capacity outer bound in parametric form so as to get insight on how to choose the number of points of the discrete part of the mixed inputs. In the weak interference

regime we will follow the same approach but the computations will be more involved because the capacity region outer bound in weak interference has three dominant faces (and not just one dominant face as in strong interference).

Outer Bound: In this regime, we express the upper bound in Proposition 6 as the convex closure of its corner points, that is

$$\mathcal{R}_{\text{out}}^{(\text{IV-C})} = \text{co} \left\{ (R_{1A}, R_{2A}) := (I_g(S), c), \right. \quad (53a)$$

$$(R_{1B}, R_{2B}) := (b - a, 2a - b), \quad (53b)$$

$$(R_{1C}, R_{2C}) := (2a - b, b - a), \quad (53c)$$

$$(R_{1D}, R_{2D}) := (c, I_g(S)) \left. \right\}, \quad (53d)$$

where

$$a := \min \left(I_g(I + S) + I_g(S) - I_g(I), 2I_g \left(I + \frac{S}{1+I} \right) \right), \quad (53e)$$

$$b := I_g \left(I + \frac{S}{1+I} \right) + I_g(S + I) + I_g(S) - I_g(I), \quad (53f)$$

$$\begin{aligned} c &:= I_g(I + S) - I_g(S) + I_g \left(I + \frac{S}{1+I} \right) - I_g(I) \\ &= I_g \left(\frac{I}{1+S} \right) + I_g \left(\frac{S}{(1+I)^2} \right) \\ &\leq I_g \left(\frac{S}{1+S} \right) + I_g \left(\frac{I}{1+I} \right) \leq \log(2). \end{aligned} \quad (53g)$$

Under the constraint $\frac{S}{1+I} \leq I$ it can be verified numerically that actually $c \leq 0.5537$ bits (rather than $c \leq 1$ bit) attained for $I = \sqrt{3} + 1$; however, for notational convenience we will use in the following $c \leq 1$ bit.

An explicit expression for $\mathcal{R}_{\text{out}}^{(\text{IV-C})}$ obtained by time-sharing between the corner points in (53) is

$$\mathcal{R}_{\text{out}}^{(\text{IV-C})} = \mathcal{R}_{2R_1+R_2}^{(\text{IV-C})} \cup \mathcal{R}_{R_1+R_2}^{(\text{IV-C})} \cup \mathcal{R}_{R_1+2R_2}^{(\text{IV-C})}, \quad \text{where} \quad (54a)$$

$$\mathcal{R}_{2R_1+R_2}^{(\text{IV-C})} = \bigcup_{t \in [0,1]} \left\{ \begin{array}{l} R_1 \leq tR_{1A} + (1-t)R_{1B} \\ R_2 \leq tR_{2A} + (1-t)R_{2B} \end{array} \right\}, \quad (54b)$$

$$\mathcal{R}_{R_1+R_2}^{(\text{IV-C})} = \bigcup_{t \in [0,1]} \left\{ \begin{array}{l} R_1 \leq tR_{1B} + (1-t)R_{1C} \\ R_2 \leq tR_{2B} + (1-t)R_{2C} \end{array} \right\}, \quad (54c)$$

$$\mathcal{R}_{R_1+2R_2}^{(\text{IV-C})} = \bigcup_{t \in [0,1]} \left\{ \begin{array}{l} R_1 \leq tR_{1C} + (1-t)R_{1D} \\ R_2 \leq tR_{2C} + (1-t)R_{2D} \end{array} \right\}. \quad (54d)$$

Because the sum-rate upper bound in (53) is in the form

$$R_1 + R_2 \leq \text{eq.}(53e) = \min(\text{eq.}(36d), \text{eq.}(36e)),$$

we will distinguish between two cases: when the constraint in (36d) is active, referred to as *Weak1*, and when the constraint in (36e) is active, referred to as *Weak2*, that is, within $l \leq S \leq l(1+l)$ we further distinguish between

$$\text{Weak1: } \frac{1+S}{1+l+\frac{S}{1+l}} \leq \frac{1+l+\frac{S}{1+l}}{1+\frac{S}{1+l}}, \quad (55)$$

$$\text{Weak2: } \frac{1+S}{1+l+\frac{S}{1+l}} > \frac{1+l+\frac{S}{1+l}}{1+\frac{S}{1+l}}. \quad (56)$$

Inner Bound: For the G-IC in weak interference the best know strategy is to send common and private messages from Gaussian codebooks, and for each of the receivers to jointly decode both common messages and the desired private message while treating the private message of the interferer as noise. Unlike in the strong and very strong interference regimes, in this case we will use the Gaussian portion of the mixed inputs by setting δ_1 and δ_2 to be non-zero. Moreover, we will vary (δ_1, δ_2) jointly with (N_1, N_2) to mimic time sharing and power control.

In this regime, we further simplify the achievable rate region in (34) from Proposition 5 as follows

$$\mathcal{R}_{\text{in}}^{(\text{IV-C})} = \bigcup_{\substack{[N_1, N_2, \delta_1, \delta_2] \in \mathbb{N}^2 \times [0, 1]^2 : \\ \max(\delta_1, \delta_2) \leq \frac{1}{1+l}}} \mathcal{R}_{\text{in}}^{(\text{IV-C})}([N_1, N_2, \delta_1, \delta_2]),$$

where

$$\mathcal{R}_{\text{in}}^{(\text{IV-C})}([N_1, N_2, \delta_1, \delta_2]) := \left\{ \begin{array}{l} 0 \leq R_1 \leq \log(N_1) + I_g(S\delta_1) - \Delta_{(57)} \\ 0 \leq R_2 \leq \log(N_2) + I_g(S\delta_2) - \Delta_{(57)} \\ \Delta_{(57)} = \frac{1}{2} \log\left(\frac{\pi e}{3}\right) + \frac{1}{2} \log\left(1 + \frac{12}{\min_{i \in [1:2]} d_{\min(S_i)}^2}\right) \end{array} \right\}, \quad (57)$$

where the received constellations S_1 and S_2 are given in (35). Note that, inspired by [23], we restricted the power splits between the continuous and discrete parts of the mixed inputs to

satisfy $\max(\delta_1, \delta_2) \leq \frac{1}{1+\gamma}$. The simplified form of the TINnoTS region with mixed inputs in (57) is obtained from (34) as follows. For the achievable rate R_1 we have

$$\begin{aligned}
R_1 &\geq \mathsf{I}_d(S_1) + \mathsf{I}_g\left(\frac{S\delta_1}{1 + \mathsf{I}\delta_2}\right) \\
&\quad - \min\left(\log(N_2), \mathsf{I}_g\left(\frac{\mathsf{I}(1 - \delta_2)}{1 + \mathsf{I}\delta_2}\right)\right) \\
&\stackrel{(a)}{=} \left[\log(N_1 N_2) - \frac{1}{2} \log\left(\frac{\pi e}{6}\right) - \frac{1}{2} \log\left(1 + \frac{12}{d_{\min(S_1)}^2}\right) \right]^+ \\
&\quad + \mathsf{I}_g\left(\frac{S\delta_1}{1 + \mathsf{I}\delta_2}\right) - \min\left(\log(N_2), \mathsf{I}_g\left(\frac{\mathsf{I}(1 - \delta_2)}{1 + \mathsf{I}\delta_2}\right)\right) \\
&\stackrel{(b)}{\geq} \log(N_1 N_2) - \frac{1}{2} \log\left(\frac{\pi e}{6}\right) - \frac{1}{2} \log\left(1 + \frac{12}{d_{\min(S_1)}^2}\right) \\
&\quad + \mathsf{I}_g\left(\frac{S\delta_1}{1 + \mathsf{I}\delta_2}\right) - \log(N_2) \\
&\stackrel{(c)}{\geq} \log(N_1) + \mathsf{I}_g\left(\frac{S\delta_1}{2}\right) + \frac{1}{2} \log(2) - \Delta_{(57)} \\
&\stackrel{(d)}{\geq} \log(N_1) + \mathsf{I}_g(S\delta_1) - \Delta_{(57)},
\end{aligned}$$

where the (in)equalities are due to: (a) because regardless of whether we use Proposition 2 or Proposition 3 to compute the minimum distance for the received sum-set constellations S_1 and S_2 in (35), these constellations always comprise $|S_1| = |S_2| = N_1 N_2$ equally likely points either exactly or almost surely; (b) because $[x]^+ \geq x$ and $\min(x, y) \leq x$; (c) because we imposed $\max(\delta_1, \delta_2) \leq \frac{1}{1+\gamma}$ and by definition of $\Delta_{(57)}$ in (57); and (d) because $\log(1 + x/2) \geq \log(1 + x) - \log(2)$. The rate expression for user 2 follows similarly.

For the evaluation of $\Delta_{(57)}$, the minimum distance of the received constellations S_1 and S_2 defined in (35) will be computed with either Proposition 2 or Proposition 3. By using Proposition 3, which is valid for all channel gains up to a set of controllable Lebesgue measure less than γ , for any $\gamma > 0$, we have

$$\begin{aligned}
&\frac{d_{\min(S_1)}^2}{12 \kappa_{\gamma, N_1, N_2}^2} \\
&\geq \frac{\min\left(\frac{(1-\delta_1)S}{N_1^2-1}, \frac{(1-\delta_2)\mathsf{I}}{N_2^2-1}, \max\left(\frac{(1-\delta_2)\mathsf{I}}{N_1^2(N_2^2-1)}, \frac{(1-\delta_1)S}{N_2^2(N_1^2-1)}\right)\right)}{1 + S\delta_1 + \mathsf{I}\delta_2}
\end{aligned} \tag{58a}$$

$$\geq \frac{(1 - \max(\delta_1, \delta_2)) \min\left(\frac{S}{N_1^2 - 1}, \frac{I}{N_2^2 - 1}, \frac{\max(S, I)}{N_1^2 N_2^2 - 1}\right)}{1 + S\delta_1 + I\delta_2} \quad (58b)$$

$$\stackrel{\text{for } I \leq S}{=} \frac{(1 - \max(\delta_1, \delta_2)) \min\left(\frac{I}{N_2^2 - 1}, \frac{S}{N_1^2 N_2^2 - 1}\right)}{1 + S\delta_1 + I\delta_2}, \quad (58c)$$

and

$$\begin{aligned} & \frac{d_{\min(S_2)}^2}{12 \kappa_{\gamma, N_1, N_2}^2} \\ & \geq \frac{1 - \max(\delta_1, \delta_2)}{1 + S\delta_2 + I\delta_1} \min\left(\frac{I}{N_1^2 - 1}, \frac{S}{N_1^2 N_2^2 - 1}\right), \end{aligned} \quad (58d)$$

where

$$\kappa_{\gamma, N_1, N_2} = \frac{\gamma/2}{1 + 1/2 \ln(\max(N_1^2, N_2^2))}. \quad (58e)$$

If instead we use Proposition 2 we have

$$\min_{i \in [1:2]} \frac{d_{\min(S_i)}^2}{12} = \min_{(i, i') \in \{(1, 2), (2, 1)\}} \frac{\min\left(\frac{(1 - \delta_i)S}{N_i^2 - 1}, \frac{(1 - \delta_{i'})I}{N_{i'}^2 - 1}\right)}{1 + S\delta_i + I\delta_{i'}}, \quad (59a)$$

which holds if

$$I(1 - \delta_{i'}) \frac{N_{i'}^2}{N_{i'}^2 - 1} \leq \frac{S(1 - \delta_i)}{N_i^2 - 1} \quad \forall (i, i') \in \{(1, 2), (2, 1)\}. \quad (59b)$$

We observe that in (57) each achievable rate is bounded by the sum of two terms: one that depends on the number of points of the discrete part of the mixed inputs, and the other that depends on the continuous part of the mixed inputs through the power splits. This is reminiscent of rate-splitting in the Han-Kobayashi achievable scheme, where each rate is written as the sum of the common-message rate and the private-message rate. The simplified Han-Kobayashi achievable region in [23] is known to achieve the outer bound in Proposition 6 to within 1/2 bit; however, to the best of our knowledge, it is not known how much information should be conveyed through the private messages and how much through the common messages for a general rate-pair (R_1, R_2) on the convex closure of the outer bound in Proposition 6 and for a general set of channel parameters. Next we will identify the (to within 1/2 bit) optimal rate splits and use the found analytical closed-form expressions for the common-message and private-message rates to come up with an educated guess for the values of the parameters of our mixed inputs.

Let $R_u = R_{u,p} + R_{u,c}$, where $R_{u,p}$ is the rate of the private message and $R_{u,c}$ is the rate of the common message for user $u \in [1 : 2]$. From the analysis of the symmetric LDA in [18,

Lemma 4], which gives the optimal gDoF region for the symmetric G-IC before Fourier-Motzkin elimination, it is not difficult to see that it is always optimal to set

$$R_{u,p} \cong \min \left(\log \left(\frac{S}{1+l} \right), \frac{R_u}{2} \right), \quad u \in [1 : 2], \quad (60)$$

where with \cong we mean equality up to an additive term that grows slower than $\log(S)$ when $S \rightarrow \infty$. We found that, with the exception of the sum-capacity for $\alpha \in (1/2, 2/3)$, the optimal ‘rate splits’ are unique and are given by (60). These ‘rate splits’ shed light on the interplay between private and common messages, which was not immediately obvious from the outer bound in (36).

In the following it will turn out to be convenient to think of the discrete part of a mixed input (contributing to the rate with the term $\log(N_i), i \in [1 : 2]$) as a ‘common message’ and of the continuous part of a mixed input (contributing to the rate with the term $\log(S\delta_i), i \in [1 : 2]$) as a ‘private message’. We shall refer to this ‘mapping’ of our TINnoTS scheme to the Han-Kobayashi scheme as the *discrete*→*common map*. Note that there is a fundamental difference between a common message in the Han-Kobayashi achievable scheme and the discrete part of the mixed input in our scheme. In our scheme the interfering signal is treated as noise while in Han-Kobayashi achievable scheme the common message is jointly decoded, albeit non-uniquely, with the intended signals at the non-intended receiver. The *discrete*→*common map* is thus just intended to provide an educated guess on how to pick the parameters of our mixed input in the following analysis. We do not claim here that the *discrete*→*common map* is the only possible way to ‘match’ our TINnoTS scheme to the Han-Kobayashi scheme. In fact, we will give an example later on where with the proposed *discrete*→*common map* we obtain a $O \left(\log \left(\frac{\ln(\min(S,l))}{\gamma} \right) \right)$ gap, but with a *discrete*→*private map* we obtain a constant gap. Although finding the smallest possible gap in each regime would be desirable, here for sake of simplicity we consistently use the *discrete*→*common map*.

With the inner and outer bounds defined, as well as the ‘rate splits’, we are ready to determine an optimal (to within a gap) choice of parameters for the mixed inputs in the weak interference regime. Next, we will focus on the regime in (55) and the regime in (56) separately and for each regime we will match each point on the closure of the outer bound in (53) with an achievable region as in (57).

D. Moderately Weak Interference, subregime Weak I

The regime of interest here is the subset of $I \leq S \leq I(1 + I)$ for which (55) holds. For convenience, we analyze the regime $I \leq S \leq 1 + I$ in Appendix D and focus next on the subset of $(1 + I) \leq S \leq I(1 + I)$ for which (55) holds. The condition $1 + I \leq S$ allows us to state $\frac{1+S}{1+I+\frac{S}{1+I}} \geq 1$ in the following.

Outer Bound Corner Points and Rate Splits: Whenever the condition in (55) holds, the outer bound in (36) is given by all the constraints in (36) except for the one in (36e) – in the symmetric case the constraints in (36c) and (36d) are the same. The corner points for the outer bound region in (54) are thus

$$\text{eq.}(36a) = \text{eq.}(36f) \Rightarrow (R_{1A}, R_{2A}) = (I_g(S), \quad (61a)$$

$$I_g\left(\frac{S}{1+I}\right) + I_g\left(I + \frac{S}{1+I}\right) - I_g(S) \Big); \quad (61b)$$

$$\text{eq.}(36f) = \text{eq.}(36c) \Rightarrow (R_{1B}, R_{2B}) = \left(I_g\left(I + \frac{S}{1+I}\right), \quad (61c)$$

$$I_g(S) + I_g\left(\frac{S}{1+I}\right) - I_g\left(I + \frac{S}{1+I}\right) \Big); \quad (61d)$$

$$\begin{aligned} \text{eq.}(36f) = \text{eq.}(36c) \Rightarrow (R_{1C}, R_{2C}) = & \left(I_g(S) + I_g\left(\frac{S}{1+I}\right) \right. \\ & \left. - I_g\left(I + \frac{S}{1+I}\right), \quad I_g\left(I + \frac{S}{1+I}\right) \right); \end{aligned} \quad (61e)$$

$$\begin{aligned} \text{eq.}(36b) = \text{eq.}(36g) \Rightarrow (R_{1D}, R_{2D}) = & \left(I_g\left(\frac{S}{1+I}\right) \right. \\ & \left. + I_g\left(I + \frac{S}{1+I}\right) - I_g(S), \quad I_g(S) \right). \end{aligned} \quad (61f)$$

As explained before, inspired by the proposed *discrete* \rightarrow *common map*, we choose to ‘split’ the rates as:

- 1) for the sum-rate face / region $\mathcal{R}_{R_1+R_2}$: we set $R_{1,p} = R_{2,p} \cong I_g\left(\frac{S}{1+I}\right)$;
- 2) for the other dominant face / region $\mathcal{R}_{2R_1+R_2}$: we set $R_{1,p} \cong I_g\left(\frac{S}{1+I}\right)$ and $R_{2,p} \cong \frac{R_2}{2}$; and
- 3) we will not explicitly consider the remaining dominant face / region $\mathcal{R}_{R_1+2R_2}$ because a gap result can be obtained by proceeding as for $\mathcal{R}_{2R_1+R_2}$ but with the role of the users swapped.

Outer Bound $\mathcal{R}_{R_1+R_2}$: With the corner point expressions in (61) we write the outer bound sum-rate face in (54c) as

$$\mathcal{R}_{R_1+R_2}^{(\text{IV-D})} = \bigcup_{t \in [0,1]} \left\{ \begin{array}{l} R_1 \leq \frac{t}{2} \log \left(\frac{1+l+\frac{S}{1+l}}{1+\frac{S}{1+l}} \right) + \frac{1-t}{2} \log \left(\frac{1+S}{1+l+\frac{S}{1+l}} \right) \\ \quad + \frac{1}{2} \log \left(1 + \frac{S}{1+l} \right) =: \mathsf{I}_g(\mathsf{S}_{1,a,t}) + \mathsf{I}_g \left(\frac{S}{1+l} \right) \\ R_2 \leq \frac{1-t}{2} \log \left(\frac{1+l+\frac{S}{1+l}}{1+\frac{S}{1+l}} \right) + \frac{t}{2} \log \left(\frac{1+S}{1+l+\frac{S}{1+l}} \right) \\ \quad + \frac{1}{2} \log \left(1 + \frac{S}{1+l} \right) =: \mathsf{I}_g(\mathsf{S}_{1,b,t}) + \mathsf{I}_g \left(\frac{S}{1+l} \right) \end{array} \right\}. \quad (62)$$

Inner Bound for $\mathcal{R}_{R_1+R_2}$: In order to approximately achieve the points in (62), we pick

$$N_1 = \mathsf{N}_d(\mathsf{S}_{1,a,t}), \quad (63a)$$

$$\mathsf{S}_{1,a,t} := \left(\frac{1+l+\frac{S}{1+l}}{1+\frac{S}{1+l}} \right)^t \left(\frac{1+S}{1+l+\frac{S}{1+l}} \right)^{1-t} - 1, \quad (63b)$$

$$N_2 = \mathsf{N}_d(\mathsf{S}_{1,b,t}), \quad (63c)$$

$$\mathsf{S}_{1,b,t} := \left(\frac{1+l+\frac{S}{1+l}}{1+\frac{S}{1+l}} \right)^{1-t} \left(\frac{1+S}{1+l+\frac{S}{1+l}} \right)^t - 1, \quad (63d)$$

$$\delta_1 = \frac{1}{1+l}, \quad (63e)$$

$$\delta_2 = \frac{1}{1+l}. \quad (63f)$$

Gap for $\mathcal{R}_{R_1+R_2}$: The gap between the outer bound region in (62) and the achievable rate region in (57) with the parameters as in (63) is

$$\begin{aligned} \Delta_{R_1} &= \mathsf{I}_g(\mathsf{S}_{1,a,t}) + \mathsf{I}_g \left(\frac{S}{1+l} \right) - \log(\mathsf{N}_d(\mathsf{S}_{1,a,t})) \\ &\quad - \mathsf{I}_g \left(\frac{S}{1+l} \right) + \Delta_{(57)} \\ &\leq \log(2) + \Delta_{(57)}, \end{aligned}$$

where the term $\log(2)$ is the “integrality gap” $\log(\mathsf{N}_d(x)) + \log(2) \geq \mathsf{I}_g(x)$; similarly, we have

$$\Delta_{R_2} \leq \log(2) + \Delta_{(57)}.$$

We are thus left with bounding $\Delta_{(57)}$ in (57), which is related to the minimum distance of the received constellations S_1 and S_2 defined in (35). In Appendix E-A we show that

$$\min_{i \in [1:2]} \frac{d_{\min(S_i)}^2}{12} \geq \kappa_{\gamma, N_1, N_2}^2 \cdot \frac{3}{8}, \quad (64)$$

where $\kappa_{\gamma, N_1, N_2}$ is given in (58e), and $\max(N_1^2, N_2^2) - 1 \leq l = \min(S, l)$. With this, the gap for this face is bounded by

$$\begin{aligned} \text{gap}_{(65)} &\leq \max(\Delta_{R_1}, \Delta_{R_2}) = \log(2) + \Delta_{(57)} \\ &\leq \frac{1}{2} \log\left(\frac{4\pi e}{3}\right) + \frac{1}{2} \log\left(1 + \frac{8}{3} \cdot \frac{1}{\kappa_{\gamma, N_1, N_2}^2}\right) \\ &\leq \text{Gap}\left(4, \frac{32}{3}\right). \end{aligned} \quad (65)$$

Gap for $\mathcal{R}_{2R_1+R_2}$: The derivation of the gap and other results for $\mathcal{R}_{2R_1+R_2}$ are delegated to Appendix E-B. The resulting gap is

$$\text{gap}_{(66)} \leq \text{Gap}(16, 45). \quad (66)$$

Overall Gap for Weak1: To conclude the proof for this sub-regime, the gap is the maximum between the gaps of the different faces and is given by

$$\text{gap}_{(67)} \leq \max(\text{gap}_{(65)}, \text{gap}_{(66)}) = \text{gap}_{(66)}. \quad (67)$$

E. Moderately Weak Interference, subregime Weak2

We focus here on the subset of $l \leq S \leq l(1+l)$ for which (56) holds.

Outer Bound Corner Points and Rate Splits: Under the condition in (56), the outer bound in (36) is given by all the constraints except for the ones in (36c) and (36d). The corner points are thus

$$\text{eq.}(36a) = \text{eq.}(36f) \Rightarrow (R_{1A}, R_{2A}) = (I_g(S), \quad (68a)$$

$$I_g\left(\frac{S}{1+l}\right) + I_g\left(l + \frac{S}{1+l}\right) - I_g(S) \Big); \quad (68b)$$

$$\begin{aligned} \text{eq.}(36f) = \text{eq.}(36e) \Rightarrow (R_{1B}, R_{2B}) &= \left(I_g\left(\frac{S}{1+l}\right) + I_g(S) \right. \\ &\quad \left. - I_g\left(l + \frac{S}{1+l}\right), \right. \end{aligned} \quad (68c)$$

$$\left. 3I_g\left(l + \frac{S}{1+l}\right) - I_g(S) - I_g\left(\frac{S}{1+l}\right) \right); \quad (68d)$$

$$\begin{aligned} \text{eq.}(36\text{e}) = \text{eq.}(36\text{g}) \Rightarrow (R_{1C}, R_{2C}) &= \left(3I_g \left(1 + \frac{S}{1+I} \right) \right. \\ &\quad \left. - I_g(S) - I_g \left(\frac{S}{1+I} \right), \right. \end{aligned} \quad (68\text{e})$$

$$\left. I_g \left(\frac{S}{1+I} \right) + I_g(S) - I_g \left(1 + \frac{S}{1+I} \right) \right); \quad (68\text{f})$$

$$\begin{aligned} \text{eq.}(36\text{b}) = \text{eq.}(36\text{g}) \Rightarrow (R_{1D}, R_{2D}) &= \left(I_g \left(\frac{S}{1+I} \right) \right. \\ &\quad \left. + I_g \left(1 + \frac{S}{1+I} \right) - I_g(S), I_g(S) \right). \end{aligned} \quad (68\text{g})$$

As explained before, inspired by the proposed *discrete* \rightarrow *common map*, we choose to ‘split’ the rates as:

- 1) for the sum-rate face / region $\mathcal{R}_{R_1+R_2}$: we set $R_{1,p} \cong \frac{R_1}{2}$ and $R_{2,p} \cong \frac{R_2}{2}$;
- 2) for the other dominant face / region $\mathcal{R}_{2R_1+R_2}$: we set $R_{1,p} \cong I_g \left(\frac{S}{1+I} \right)$ and $R_{2,p} \cong \frac{R_2}{2}$; and
- 3) we will not explicitly consider the remaining dominant face / region $\mathcal{R}_{R_1+2R_2}$ because a gap result can be obtained by proceeding as for $\mathcal{R}_{2R_1+R_2}$ but with the role of the users swapped.

Outer Bound $\mathcal{R}_{R_1+R_2}$: With the corner point expressions in (68) we write the outer bound sum-rate face in (54c) as

$$\mathcal{R}_{R_1+R_2}^{(\text{IV-E})} = \bigcup_{t \in [0,1]} \left\{ \begin{array}{l} R_1 \leq \frac{1-t}{2} \log \left(\frac{(1+\frac{S}{1+I})(1+S)}{1+I+\frac{S}{1+I}} \right) \\ \quad + \frac{t}{2} \log \left(\frac{(1+I+\frac{S}{1+I})^3}{(1+\frac{S}{1+I})(1+S)} \right) =: 2 \cdot I_g(S_{3,a,t}) \\ R_1 \leq \frac{t}{2} \log \left(\frac{(1+\frac{S}{1+I})(1+S)}{1+I+\frac{S}{1+I}} \right) \\ \quad + \frac{1-t}{2} \log \left(\frac{(1+I+\frac{S}{1+I})^3}{(1+\frac{S}{1+I})(1+S)} \right) =: 2 \cdot I_g(S_{3,b,t}) \end{array} \right\}. \quad (69)$$

Inner Bound for $\mathcal{R}_{R_1+R_2}$: In order to approximately achieve the points in $\mathcal{R}_{R_1+R_2}^{(\text{IV-E})}$ in (69) we pick

$$\begin{aligned} N_1 &= N_d(S_{3,a,t}), \quad (70\text{a}) \\ S_{3,a,t} &:= \left(\frac{(1+\frac{S}{1+I})(1+S)}{1+I+\frac{S}{1+I}} \right)^{\frac{1-t}{2}} \left(\frac{(1+I+\frac{S}{1+I})^3}{(1+\frac{S}{1+I})(1+S)} \right)^{\frac{t}{2}} \end{aligned}$$

$$-1, \quad (70b)$$

$$N_2 = N_d(S_{3,b,t}), \quad (70c)$$

$$S_{3,b,t} := \left(\frac{(1 + \frac{S}{1+I})(1+S)}{1+I + \frac{S}{1+I}} \right)^{\frac{t}{2}} \left(\frac{(1+I + \frac{S}{1+I})^3}{(1 + \frac{S}{1+I})(1+S)} \right)^{\frac{1-t}{2}} - 1, \quad (70d)$$

$$\delta_1 : I_g(S\delta_1) = I_g(S_{3,a,t}) \iff \delta_1 = \frac{S_{3,a,t}}{S}, \quad (70e)$$

$$\delta_2 : I_g(S\delta_2) = I_g(S_{3,b,t}) \iff \delta_2 = \frac{S_{3,b,t}}{S}, \quad (70f)$$

where

$$\max(\delta_1, \delta_2) = \frac{\max(S_{3,a,t}, S_{3,b,t})}{S} \leq \frac{1}{1+I},$$

as required for the achievable rate in (57); the proof can be found in Appendix F-A, eq.(137).

Gap for $\mathcal{R}_{R_1+R_2}$: The gap between the outer bound region in (69) and the achievable rate in (57) with the parameters in (152) is

$$\begin{aligned} \Delta_{R_1} &= 2I_g(S_{3,a,t}) - \log(N_d(S_{3,a,t})) - I_g(S_{3,a,t}) + \Delta_{(57)} \\ &\leq \log(2) + \Delta_{(57)}, \end{aligned}$$

and similarly

$$\Delta_{R_2} \leq \log(2) + \Delta_{(57)}.$$

We are then left with bounding $\Delta_{(57)}$, which depends on minimum distances of the received sum-set constellations. In Appendix F-A we show

$$\min_{i \in [1:2]} \frac{d_{\min}^2(S_i)}{12} \geq \kappa_{\gamma, N_1, N_2}^2 \cdot \frac{1}{24}, \quad (71)$$

where $\kappa_{\gamma, N_1, N_2}$ is given in (58e), and $\max(N_1^2, N_2^2) - 1 \leq I = \min(S, I)$. With this, we finally get that the gap for this face is bounded by

$$\begin{aligned} \text{gap}_{(72)} &\leq \max(\Delta_{R_1}, \Delta_{R_2}) = \log(2) + \Delta_{(57)} \\ &\leq \text{Gap}(4, 96) \end{aligned} \quad (72)$$

Gap for $\mathcal{R}_{2R_1+R_2}$: The derivation of the gap and other results for $\mathcal{R}_{2R_1+R_2}$ are delegated to Appendix F-B. The resulting gap is

$$\text{gap}_{(73)} \leq \text{Gap}(16, 32). \quad (73)$$

Overall Gap for Weak2: To conclude the proof for this sub-regime, the gap is the maximum between the gaps of the different faces and is given by

$$\text{gap}_{(74)} = \max(\text{gap}_{(72)}, \text{gap}_{(73)}) = \text{gap}_{(73)}. \quad (74)$$

Another Overall Gap for Weak2: The choice of the mixed input parameters according to the *discrete*→*common map* in (70) and in (72) led to the $O\left(\log\left(\frac{\ln(\min(S, I))}{\gamma}\right)\right)$ gap in (74). This is so because we used Proposition 3 to bound the minimum distance. A interesting question is whether Proposition 2 could be used, possibly with a different choice of mixed input parameters.

With a gDoF-type analysis, one can show that it is possible to verify the condition in Proposition 2 with the proposed choice of parameters in (70) but not with the input parameters in (72). So, in this regime we are motivated to look at the *discrete*→*private map* as we hope to get a constant gap result for the whole region. In Appendix G we show that in this regime it is possible to use Proposition 2 and the *discrete*→*private map* to get a constant gap, namely,

$$\text{gap}_{(75)} = \text{Gap}\left(\frac{608}{9}\right) \approx 3.79 \text{ bits}. \quad (75)$$

F. Very weak interference, i.e., $I(1 + I) \leq S$

In this regime the capacity of the classical G-IC is achieved to within a constant gap by Gaussian inputs, treating interference as noise and power control. This strategy is compatible without the TINnoTS scheme (i.e., set $N_1 = N_2 = 1$ and vary δ_1 and δ_2), so the gap of

$$\text{gap}_{(76)} \leq 1/2 \text{ bit}, \quad (76)$$

as shown in [23] holds.

This concludes the proof of Theorem 7. ■

V. GAP FOR SOME ASYMMETRIC CHANNELS

In this Section we generalize the gap result of Theorem 7 to some general asymmetric settings.

Theorem 8. *For the general G-IC, except for the regime*

$$\frac{|h_{22}|^2}{1 + |h_{21}|^2} < |h_{12}|^2 < \frac{|h_{22}|^2}{1 + |h_{21}|^2}(1 + |h_{11}|^2), \quad (77a)$$

$$\frac{|h_{11}|^2}{1 + |h_{12}|^2} < |h_{21}|^2 < \frac{|h_{11}|^2}{1 + |h_{12}|^2}(1 + |h_{22}|^2), \quad (77b)$$

akin to the moderately weak interference regime for the symmetric setting, the TINnoTS achievable region and the outer bound in Proposition 6 are to within an additive gap that is either constant or of the order $O\left(\log \frac{\ln(\max(|h_{11}|^2, |h_{22}|^2))}{\gamma}\right)$.

Remark 7 (Why is the regime in (77) excluded?). The regime identified in (77) involves numerous special cases, whose analysis gets very tedious. We do however strongly believe that our gap result generalizes to this regime as well, by using similar arguments to those developed so far. We note that the analysis in the rest of this section for the general asymmetric setting (which is characterized by four channel parameters) is restricted to those cases where it suffices to consider at most one rate split (thus reducing the number of parameters to be optimize for the mixed inputs) and for which the approximately optimal rate region does not require bounds on $2R_1 + R_2$ or $R_1 + 2R_2$ (thus reducing the achievability to the sum-capacity dominant face only).

Proof: We shall treat different regimes separately in the rest of the section.

A. Very Strong Interference

In the general asymmetric case, the very strong interference regime is the regime in which a receiver can decode the interfering message while treating its intended signal as noise at a higher rate than the intended receiver in the absence of interference; this is the case when the channel gains satisfy [30]

$$|h_{11}|^2(1 + |h_{22}|^2) \leq |h_{21}|^2, \quad (78a)$$

$$|h_{22}|^2(1 + |h_{11}|^2) \leq |h_{12}|^2. \quad (78b)$$

Outer Bound: The capacity region of the classical G-IC in very strong interference coincides with that of two interference-free point-to-point links given by

$$\mathcal{R}_{\text{out}}^{(\text{V-A})} = \left\{ \begin{array}{l} 0 \leq R_1 \leq \log(|h_{11}|^2) \\ 0 \leq R_2 \leq \log(|h_{22}|^2) \end{array} \right\}. \quad (79)$$

Inner Bound: The outer bound in (79) can be matched to within a constant gap by our TINnoTS scheme by choosing, similarly to the symmetric case discussed in Section IV-A, the mixed inputs in (33) with

$$N_1 = N_d(\beta|h_{11}|^2) : N_1^2 - 1 \leq \beta|h_{11}|^2, \quad (80a)$$

$$N_2 = \mathbf{N}_d \left(\beta |h_{22}|^2 \right) : N_2^2 - 1 \leq \beta |h_{22}|^2, \quad (80b)$$

$$\delta_1 = 0, \quad (80c)$$

$$\delta_2 = 0, \quad (80d)$$

for some $\beta \leq 1$. The reason for the factor β in (80) will be clear shortly (in Appendix G we use $\beta = 3/4$ for similar reasons; we could have used here $\beta = 3/4$ as well, but we will find next a value that gives a smaller gap).

We next show that Proposition 2 is applicable for the choice of mixed input parameters as in (80). In particular, we aim to show that

$$N_1 |h_{11}| d_{\min(X_1)} \leq |h_{12}| d_{\min(X_2)} \quad (81a)$$

(for the received sum-set constellation at receiver 1)

$$N_2 |h_{22}| d_{\min(X_2)} \leq |h_{21}| d_{\min(X_1)} \quad (81b)$$

(for the received sum-set constellation at receiver 2),

or equivalently that

$$\frac{N_1^2}{N_1^2 - 1} \cdot \frac{|h_{11}|^2}{1 + |h_{11}|^2} \cdot \frac{N_2^2 - 1}{|h_{22}|^2} \leq \frac{|h_{12}|^2}{|h_{22}|^2 (1 + |h_{11}|^2)}, \quad (82a)$$

$$\frac{N_2^2}{N_2^2 - 1} \cdot \frac{|h_{22}|^2}{1 + |h_{22}|^2} \cdot \frac{N_1^2 - 1}{|h_{11}|^2} \leq \frac{|h_{21}|^2}{|h_{11}|^2 (1 + |h_{22}|^2)}. \quad (82b)$$

The condition in (82) is verified, given the channel gain relationship in (78), if

$$\frac{N_1^2}{N_1^2 - 1} \cdot \frac{|h_{11}|^2}{1 + |h_{11}|^2} \cdot \frac{N_2^2 - 1}{|h_{22}|^2} \leq 1, \quad (83a)$$

$$\frac{N_2^2}{N_2^2 - 1} \cdot \frac{|h_{22}|^2}{1 + |h_{22}|^2} \cdot \frac{N_1^2 - 1}{|h_{11}|^2} \leq 1. \quad (83b)$$

It can be easily seen that $\beta = 0.8277$ satisfies (83) whenever $2 \leq \min(N_1, N_2)$. For the found β we therefore have that the received constellations have $|S_1| = |S_2| = N_1 N_2$ equally likely points and minimum distance

$$\min_{i \in [1:2]} \frac{d_{\min(S_i)}^2}{12} = \min \left(\frac{|h_{11}|^2}{N_1^2 - 1}, \frac{|h_{22}|^2}{N_2^2 - 1} \right) \geq \frac{1}{\beta}. \quad (84)$$

Thus, by following similar steps as in Section IV-A, the achievable region becomes

$$\mathcal{R}_{\text{in}}^{(\text{V-A})} = \left\{ \begin{array}{l} 0 \leq R_1 \leq r_1 \\ 0 \leq R_2 \leq r_2 \end{array} \right\} \text{ such that} \quad (85a)$$

$$\begin{aligned}
r_1 &\geq \mathsf{I}_d(S_1) - \min(\log(N_2), \mathsf{I}_g(|h_{12}|^2)) \\
&\geq \log(N_1) - \Delta_{(85)},
\end{aligned} \tag{85b}$$

$$\begin{aligned}
r_2 &\geq \mathsf{I}_d(S_2) - \min(\log(N_1), \mathsf{I}_g(|h_{21}|^2)) \\
&\geq \log(N_2) - \Delta_{(85)},
\end{aligned} \tag{85c}$$

$$\begin{aligned}
\Delta_{(85)} &\leq \frac{1}{2} \log\left(\frac{2\pi e}{12}\right) + \frac{1}{2} \log\left(1 + \frac{12}{\min_{i \in [1:2]} d_{\min(S_i)}^2}\right) \\
&\leq \frac{1}{2} \log\left(\frac{2\pi e}{12}(\beta + 1)\right).
\end{aligned} \tag{85d}$$

Gap: We can easily see, by comparing the inner bound in (85) with the outer bound in (79), that for the general asymmetric G-IC in very strong interference the TINnoTS region is optimal to within

$$\begin{aligned}
\text{gap}_{(86)} &\leq \Delta_{(85)} + \log(2) + \frac{1}{2} \log\left(\frac{1}{\beta}\right) \\
&\leq \frac{1}{2} \log\left(\frac{2\pi e}{3} \frac{1+\beta}{\beta}\right) \\
&= \text{Gap}\left(2\frac{1+\beta}{\beta}, 0\right) \stackrel{\beta=0.8277}{\approx} 1.8260 \text{ bits},
\end{aligned} \tag{86}$$

where the term $\log(2)$ is the integrality gap and the term $\frac{1}{2} \log\left(\frac{1}{\beta}\right)$ because of the reduced number of points in (80).

B. Strong (but not Very Strong) Interference

For the general asymmetric case, the strong interference regime is defined as

$$|h_{21}|^2 \geq |h_{11}|^2, \tag{87a}$$

$$|h_{12}|^2 \geq |h_{22}|^2. \tag{87b}$$

The strong (but not very strong) interference regime is the set of channel gains that satisfy the condition in (87) but not the condition in (78).

Outer Bound: The capacity region of the general G-IC in the strong interference regime is given by the ‘compound MAC’ region

$$\mathcal{R}_{\text{out}}^{(\text{V-B})} =$$

$$\left\{ \begin{array}{l} 0 \leq R_1 \leq \mathsf{I}_{\mathbf{g}}(|h_{11}|^2) \\ 0 \leq R_2 \leq \mathsf{I}_{\mathbf{g}}(|h_{22}|^2) \\ R_1 + R_2 \leq \mathsf{I}_{\mathbf{g}}(\min(|h_{11}|^2 + |h_{12}|^2, |h_{22}|^2 + |h_{21}|^2)) \end{array} \right\}. \quad (88)$$

Inner Bound: The outer bound in (88) can be matched to within a gap by our TINnoTS scheme by choosing, similarly to the symmetric case discussed in detail in Section IV-B, the parameters of the mixed inputs as

$$\begin{aligned} N_1 &= \mathbf{N}_{\mathbf{d}}(S_{5,a,t}), \\ S_{5,a,t} &= (1 + |h_{11}|^2)^{1-t} \\ &\quad \cdot \left(\frac{1 + \min(|h_{11}|^2 + |h_{12}|^2, |h_{22}|^2 + |h_{21}|^2)}{1 + |h_{22}|^2} \right)^t - 1 \\ &\leq |h_{11}|^2, \end{aligned} \quad (89a)$$

$$\begin{aligned} N_2 &= \mathbf{N}_{\mathbf{d}}(S_{5,b,t}), \\ S_{5,b,t} &= (1 + |h_{22}|^2)^t \\ &\quad \cdot \left(\frac{1 + \min(|h_{11}|^2 + |h_{12}|^2, |h_{22}|^2 + |h_{21}|^2)}{1 + |h_{11}|^2} \right)^{1-t} - 1 \\ &\leq |h_{22}|^2, \end{aligned} \quad (89b)$$

$$\delta_1 = 0, \quad (89c)$$

$$\delta_2 = 0, \quad (89d)$$

where the upper bounds on $S_{5,a,t}$ and $S_{5,b,t}$ are a consequence of not being in very strong interference, i.e.,

$$\begin{aligned} &\min(1 + |h_{11}|^2 + |h_{12}|^2, 1 + |h_{22}|^2 + |h_{21}|^2) \\ &\leq (1 + |h_{11}|^2)(1 + |h_{22}|^2). \end{aligned}$$

Next, by using Proposition 3 we have

$$\begin{aligned} \frac{d_{\min(S_1)}^2}{12 \kappa_{\gamma, N_1, N_2}^2} &\geq \min \left(\frac{|h_{11}|^2}{N_1^2 - 1}, \frac{|h_{12}|^2}{N_2^2 - 1}, \right. \\ &\quad \left. \max \left(\frac{|h_{12}|^2}{N_1^2(N_2^2 - 1)}, \frac{|h_{11}|^2}{N_2^2(N_1^2 - 1)} \right) \right), \\ \frac{d_{\min(S_2)}^2}{12 \kappa_{\gamma, N_1, N_2}^2} &\geq \min \left(\frac{|h_{21}|^2}{N_1^2 - 1}, \frac{|h_{22}|^2}{N_2^2 - 1}, \right. \end{aligned} \quad (90a)$$

$$\max \left(\frac{|h_{21}|^2}{N_1^2(N_2^2 - 1)}, \frac{|h_{22}|^2}{N_2^2(N_1^2 - 1)} \right), \quad (90b)$$

where the bounds in (90) hold up to a set of measure γ and where $\kappa_{\gamma, N_1, N_2}$ is defined in (58e).

By recalling the channel gain relationship, by noting that

$$N_1^2 N_2^2 - 1 \leq \min (|h_{11}|^2 + |h_{12}|^2, |h_{22}|^2 + |h_{21}|^2)$$

and by combining the two bounds in (90) we get

$$\begin{aligned} & \min_{i \in [1:2]} \frac{d_{\min(S_i)}^2}{12 \kappa_{\gamma, N_1, N_2}^2} \\ & \geq \min \left(1, \frac{\max(|h_{11}|^2, |h_{12}|^2)}{|h_{11}|^2 + |h_{12}|^2}, \frac{\max(|h_{21}|^2, |h_{22}|^2)}{|h_{22}|^2 + |h_{21}|^2} \right) \geq \frac{1}{2}. \end{aligned}$$

Gap: By following the same reasoning and bounding steps as we did for the symmetric case in Section IV-B, we get that the proposed achievable scheme is optimal to within a gap of

$$\text{gap}_{(91)} \leq \text{Gap}(2, 8). \quad (91)$$

C. Mixed Interference

The mixed interference regime occurs when one receiver experiences strong interference while the other experiences weak interference. This regime does not appear in the symmetric case, where both receiver are either in strong interference or in weak interference. The mixed interference is defined as [23]

$$\text{either } \{ |h_{21}|^2 \geq |h_{11}|^2, |h_{12}|^2 \leq |h_{22}|^2 \}, \quad (92a)$$

$$\text{or } \{ |h_{21}|^2 \leq |h_{11}|^2, |h_{12}|^2 \geq |h_{22}|^2 \}. \quad (92b)$$

In this Section we shall only focus on the sub-regime

$$|h_{21}|^2 \geq \frac{|h_{11}|^2}{1 + |h_{12}|^2} (1 + |h_{22}|^2), \quad |h_{12}|^2 \leq |h_{22}|^2, \quad (93)$$

for which the rate region, as we shall see, does not require bounds on $2R_1 + R_2$ or $R_1 + 2R_2$.

The regime $|h_{12}|^2 \geq \frac{|h_{22}|^2}{1 + |h_{21}|^2} (1 + |h_{11}|^2)$, $|h_{21}|^2 \leq |h_{11}|^2$ can be analyzed similarly by swapping the role of the users.

Outer Bound: An outer bound to the capacity region of the general G-IC when (93) holds is given by the ‘Z-channel’ outer bound [37]

$$\mathcal{R}_{\text{out}}^{(\text{V-C})} = \left\{ \begin{array}{l} 0 \leq R_1 \leq \mathsf{I}_{\mathbf{g}}(|h_{11}|^2) \\ 0 \leq R_2 \leq \mathsf{I}_{\mathbf{g}}(|h_{22}|^2) \\ R_1 + R_2 \leq \mathsf{I}_{\mathbf{g}}(|h_{22}|^2) + \mathsf{I}_{\mathbf{g}}\left(\frac{|h_{11}|^2}{1+|h_{12}|^2}\right) \end{array} \right\} = \bigcup_{t \in [0,1]} \left\{ \begin{array}{l} 0 \leq R_1 \leq (1-t)\mathsf{I}_{\mathbf{g}}(|h_{11}|^2) + t\mathsf{I}_{\mathbf{g}}\left(\frac{|h_{11}|^2}{1+|h_{12}|^2}\right) \\ \quad = \mathsf{I}_{\mathbf{g}}(\mathsf{S}_{6,a,t}) \\ 0 \leq R_2 \leq (1-t)(\mathsf{I}_{\mathbf{g}}(|h_{22}|^2) - \mathsf{I}_{\mathbf{g}}(|h_{11}|^2) \\ \quad + \mathsf{I}_{\mathbf{g}}\left(\frac{|h_{11}|^2}{1+|h_{12}|^2}\right)) + t\mathsf{I}_{\mathbf{g}}(|h_{22}|^2) \\ \quad = \mathsf{I}_{\mathbf{g}}(\mathsf{S}_{6,b,t}) + \frac{1}{2} \log\left(\frac{1+|h_{22}|^2}{1+|h_{12}|^2}\right) \end{array} \right\}. \quad (94)$$

Inner Bound: The shape of the outer bound in (94) suggests that a matching, to within a gap, inner region could be found by following steps similar to those used for the analysis of the strong interference regime (i.e., parameterize the points on the dominate sum-capacity face). The difference between this sub-regime and the strong interference regime is that here R_2 should be a combination of common and private rates because receiver 1 experiences weak interference (while receiver 2 experiences strong interference). Note that the interfering channel gain at receiver 2, h_{21} , does not appear in the outer bound in (94). We therefore set

$$N_1 = \mathsf{N}_{\text{d}}(\mathsf{S}_{6,a,t}) : \quad (95a)$$

$$\mathsf{S}_{6,a,t} = \left(1 + \frac{|h_{11}|^2}{1+|h_{12}|^2}\right)^t (1+|h_{11}|^2)^{1-t} - 1 \leq |h_{11}|^2, \quad (95b)$$

$$N_2 = \mathsf{N}_{\text{d}}(\mathsf{S}_{6,b,t}) : \quad (95c)$$

$$\mathsf{S}_{6,b,t} = (1+|h_{12}|^2)^t \left(1 + \frac{|h_{12}|^2}{1+|h_{11}|^2}\right)^{1-t} - 1 \leq |h_{12}|^2, \quad (95d)$$

$$\delta_1 = 0, \quad (95e)$$

$$\delta_2 = \frac{1}{1+|h_{12}|^2}, \quad (95f)$$

in the achievable region in Proposition 5, which becomes

$$\mathcal{R}_{\text{in}}^{(96)} = \left\{ \begin{array}{l} 0 \leq R_1 \leq \log(N_1) - \Delta_{(96)} \\ 0 \leq R_2 \leq \log(N_2) + \mathsf{I}_{\mathbf{g}}\left(\frac{|h_{22}|^2}{1+|h_{12}|^2}\right) - \Delta_{(96)} \end{array} \right\}, \quad (96a)$$

$$\Delta_{(96)} = \frac{1}{2} \log\left(\frac{2\pi e}{12}\right) + \frac{1}{2} \log\left(1 + \frac{12}{\min_{i \in [1:2]} d_{\min(S_i)}^2}\right), \quad (96b)$$

$$S_1 = \frac{h_{11}X_{1D} + \sqrt{\frac{|h_{12}|^2}{1+|h_{12}|^2}}h_{12}X_{2D}}{\sqrt{1 + \frac{|h_{12}|^2}{1+|h_{12}|^2}}}, \quad (96c)$$

$$S_2 = \frac{h_{21}X_{1D} + \sqrt{\frac{|h_{12}|^2}{1+|h_{12}|^2}}h_{22}X_{2D}}{\sqrt{1 + |h_{22}|^2 \frac{1}{1+|h_{12}|^2}}}. \quad (96d)$$

Next, by using Proposition 3, we bound the minimum distance of the received constellations S_1 and S_2 as

$$\begin{aligned} \frac{d_{\min(S_1)}^2}{12 \kappa_{\gamma, N_1, N_2}^2} &\geq \frac{1}{1 + |h_{12}|^2 \delta_2} \min \left(\frac{|h_{11}|^2}{N_1^2 - 1}, \frac{(1 - \delta_2)|h_{12}|^2}{N_2^2 - 1}, \right. \\ &\quad \left. \max \left(\frac{(1 - \delta_2)|h_{12}|^2}{N_1^2(N_2^2 - 1)}, \frac{|h_{11}|^2}{N_2^2(N_1^2 - 1)} \right) \right) \\ &\stackrel{(a)}{\geq} \frac{1}{1 + |h_{12}|^2 \delta_2} \min(1, (1 - \delta_2), \\ &\quad \frac{\max((1 - \delta_2)|h_{12}|^2, |h_{11}|^2)}{N_1^2 N_2^2 - 1}) \\ &\geq \frac{1 - \delta_2}{1 + |h_{12}|^2 \delta_2} \min \left(1, \frac{\max(|h_{12}|^2, |h_{11}|^2)}{N_1^2 N_2^2 - 1} \right) \\ &\stackrel{(b)}{\geq} \frac{|h_{12}|^2}{2 + |h_{12}|^2} \min \left(1, \frac{\max(|h_{12}|^2, |h_{11}|^2)}{|h_{12}|^2 + |h_{11}|^2} \right) \\ &\stackrel{(c)}{\geq} \frac{1}{3} \min \left(1, \frac{1}{2} \right) = \frac{1}{6}, \end{aligned}$$

where the inequalities follow since: (a) by using the bounds in (95a) and (95c); (b) because $N_1^2 N_2^2 - 1 \leq |h_{12}|^2 + |h_{11}|^2$ from (95a) and (95c); and (c) by assuming $|h_{12}|^2 \geq 1$. Similarly we have that

$$\begin{aligned} \frac{d_{\min(S_2)}^2}{12 \kappa_{\gamma, N_1, N_2}^2} &\geq \frac{1}{1 + |h_{22}|^2 \delta_2} \min \left(\frac{|h_{21}|^2}{N_1^2 - 1}, \frac{(1 - \delta_2)|h_{22}|^2}{N_2^2 - 1}, \right. \\ &\quad \left. \max \left(\frac{|h_{21}|^2}{N_2^2(N_1^2 - 1)}, \frac{(1 - \delta_2)|h_{22}|^2}{N_1^2(N_2^2 - 1)} \right) \right) \\ &\stackrel{(a)}{\geq} \frac{1 - \delta_2}{1 + |h_{22}|^2 \delta_2} \min \left(\frac{|h_{21}|^2}{|h_{11}|^2}, \frac{|h_{22}|^2}{|h_{12}|^2}, \frac{\max(|h_{21}|^2, |h_{22}|^2)}{|h_{12}|^2 + |h_{11}|^2} \right) \\ &\stackrel{(b)}{\geq} \frac{1 - \delta_2}{1 + |h_{22}|^2 \delta_2} \min \left(\frac{1 + |h_{22}|^2}{1 + |h_{12}|^2}, \frac{|h_{22}|^2}{|h_{12}|^2}, \right. \\ &\quad \left. \frac{\max \left(|h_{11}|^2 \frac{1 + |h_{22}|^2}{1 + |h_{12}|^2}, |h_{12}|^2 \frac{|h_{22}|^2}{|h_{12}|^2} \right)}{|h_{12}|^2 + |h_{11}|^2} \right) \\ &\stackrel{(b)}{\geq} \frac{1 - \delta_2}{1 + |h_{22}|^2 \delta_2} \frac{1 + |h_{22}|^2}{1 + |h_{12}|^2} \frac{1}{2} \end{aligned}$$

$$\begin{aligned}
&= \frac{1 + |h_{22}|^2}{1 + |h_{12}|^2 + |h_{22}|^2} \frac{|h_{12}|^2}{1 + |h_{12}|^2} \frac{1}{2} \\
&\stackrel{(c)}{\geq} \frac{2}{3} \frac{1}{2} \frac{1}{2} = \frac{1}{6},
\end{aligned}$$

where the inequalities follow since: (a) by using the bounds in (95a) and (95c) and because $N_1^2 N_2^2 - 1 \leq |h_{12}|^2 + |h_{11}|^2$; (b) by the channel gain relationship in (93); and (c) by assuming $1 \leq |h_{12}|^2$ and since by assumption of this regime $|h_{12}|^2 \leq |h_{22}|^2$. Note that the assumption $1 \leq |h_{12}|^2$ is without loss of generality since if $|h_{12}|^2 < 1$ (i.e., interference below the noise floor of the receiver) then TIN with Gaussian inputs achieves the capacity outer bound (in this case essentially two interference-free point-to-point links) to within 1/2 bit.

This shows that

$$\min_{i \in [1:2]} \frac{d_{\min}^2(S_i)}{12} \geq \kappa_{\gamma, N_1, N_2}^2 \cdot \frac{1}{6}, \quad (97)$$

up to an outage set of measure no more than γ , where γ affects $\kappa_{\gamma, N_1, N_2}$.

Gap: By following the same reasoning and bounding steps as we did for the symmetric case, we get that the proposed achievable scheme is optimal to within a gap of

$$\begin{aligned}
\Delta_{R_1} &\leq \mathsf{I}_{\mathbf{g}}(\mathbf{S}_{6,a,t}) - \log(\mathsf{N}_{\mathbf{d}}(\mathbf{S}_{6,a,t})) + \Delta_{(96)} \\
&\leq \Delta_{(96)} + \log(2), \\
\Delta_{R_2} &\leq \mathsf{I}_{\mathbf{g}}(\mathbf{S}_{6,b,t}) + \frac{1}{2} \log \left(\frac{1 + |h_{22}|^2}{1 + |h_{12}|^2} \right) - \log(\mathsf{N}_{\mathbf{d}}(\mathbf{S}_{6,b,t})) \\
&\quad - \mathsf{I}_{\mathbf{g}} \left(\frac{|h_{22}|^2}{1 + |h_{12}|^2} \right) + \Delta_{(96)} \\
&\leq \Delta_{(96)} + \log(2),
\end{aligned}$$

where we used the fact that $\log(\mathsf{N}_{\mathbf{d}}(x)) \geq \mathsf{I}_{\mathbf{g}}(x) - \log(2)$. By including the minimum distance bound in (97) into the expression for $\Delta_{(96)}$ in (96b), and by noticing that $\max(N_1^2, N_2^2) - 1 \leq \max(|h_{11}|^2, |h_{12}|^2) \leq \max(|h_{11}|^2, |h_{22}|^2)$ by the channel gain relationship in (93), we finally get

$$\begin{aligned}
\text{gap}_{(98)} &\leq \frac{1}{2} \log \left(\frac{2\pi e}{3} \right) + \frac{1}{2} \log \left(1 + \frac{6}{\kappa_{\gamma, N_1, N_2}^2} \right), \\
&\leq \text{Gap}(2, 24).
\end{aligned} \quad (98)$$

D. Weak Interference

For the general asymmetric G-IC, the weak interference is defined as

$$|h_{21}|^2 \leq |h_{11}|^2, \quad (99a)$$

$$|h_{12}|^2 \leq |h_{22}|^2, \quad (99b)$$

which involves numerous special cases whose analysis gets very tedious and is outside of the scope of this paper – see also Remark 7.

E. Very Weak Interference

The very weak interference regime characterized in as [11] is defined as

$$|h_{12}|^2 \leq \frac{|h_{22}|^2}{1 + |h_{21}|^2}, \quad (100a)$$

$$|h_{21}|^2 \leq \frac{|h_{11}|^2}{1 + |h_{12}|^2}. \quad (100b)$$

In this regime, the outer bound to the capacity region of the classical G-IC is

$$\mathcal{R}_{\text{out}}^{(\text{V-E})} = \left\{ \begin{array}{l} R_1 \leq \log(|h_{11}|^2) \\ R_2 \leq \log(|h_{22}|^2) \\ R_1 + R_2 \leq \log\left(|h_{12}|^2 + \frac{|h_{11}|^2}{1 + |h_{21}|^2}\right) \\ \quad + \log\left(|h_{21}|^2 + \frac{|h_{22}|^2}{1 + |h_{12}|^2}\right) \end{array} \right\}, \quad (101)$$

and is achievable to within 1/2 bit by Gaussian inputs with power control and TIN. Since the optimal strategy for the classical G-IC is compatible with our TINnoTS with mixed inputs, we conclude that a mixed-input is optimal to within 1/2 bit in this regime.

This concludes the proof of Theorem 8. ■

VI. TINnoTS IS GDOF OPTIMAL

In this section we show one of the consequences of Theorem 7, namely that TINnoTS is gDoF optimal almost surely. The notion of gDoF has been introduced in [23] and has become an important metric that sheds lights on the behavior of the capacity when exact capacity results are not available. The gDoF region is the set

$$\mathcal{D} := \left\{ (d_1, d_2) \in \mathbb{R}_+^2 : d_i := \lim_{\substack{S \rightarrow \infty \\ S^\alpha}} \frac{R_i}{\frac{1}{2} \log(1 + S)} \right. \\ \left. i \in [1 : 2], (R_1, R_2) \text{ is achievable} \right\}. \quad (102)$$

The $O\left(\log\left(\frac{\ln(\min(S, I))}{\gamma}\right)\right)$ additive gap result of Theorem 7 implies that:

Theorem 9. *For the symmetric G-IC the TINnoTS achievable scheme with mixed inputs is gDoF optimal for all channel gains up to a set of zero measure. The optimal inputs are given in Table I.*

Proof: We must show that as $S \rightarrow \infty$ the gap between the TINnoTS inner bound and outer bound in Proposition 6, normalized by $I_g(S)$, goes to zero almost everywhere.

In the proof of Theorem 7 we showed that for the very strong, the weak2 and the very weak interference regimes the gap between inner and outer bounds is $O(1)$ everywhere. Therefore, since $\lim_{S \rightarrow \infty} \frac{O(1)}{I_g(S)} = 0$, the result follows.

For the strong and weak1 interference regimes the gap is of the form $O\left(\log\left(\frac{\ln \min(S, S^\alpha)}{\gamma}\right)\right)$ for any $\gamma \in (0, 1]$. Therefore, by choosing γ to be

$$\gamma(S) := \frac{1}{(\log \min(S, S^\alpha))^p}, \text{ for any } p > 0 \text{ independent of } S,$$

we have that $\lim_{S \rightarrow \infty} \frac{O\left(\log\left(\frac{\ln \min(S, S^\alpha)}{\gamma}\right)\right)}{I_g(S)} = 0$ and the measure of the outage set $\gamma(S)$ vanishes as $S \rightarrow \infty$. This concludes the proof. ■

Remark 8. *We note that in [38], the authors showed that discrete-continuous mixtures are strictly sub-optimal in the DoF expression [38, Theorem 4] for $K > 2$ user. However, this does not mean that for an equivalent (but different) expression of the DoF continuous and discrete mixtures are not optimal. For example, in [39] it was shown that Gaussian inputs do not maximize the multi-letter capacity expression for Gaussian multiple access channels. However, in an equivalent single letter expression of the capacity Gaussian inputs are optimal. Something similar occurs in the context of K -user interference channel. On the one hand, in [38] it was shown that discrete-continuous mixtures are not optimal when used in a particular capacity expression (involving information dimension). On the other hand, from [28] we know that taking a discrete distribution that depends on the SNR (i.e., the number of points scales with SNR) and using this in another expression, one can achieve $\text{DoF} = \frac{K}{2}$ a.e.*

Moreover, the result of [38], does not imply that discrete-continuous mixtures can not be capacity achieving for some parameter regimes. In particular in [11] authors showed that Gaussian inputs are gDoF optimal (for $K \geq 2$) in the so called weak interference regime.

VII. TOTALLY ASYNCHRONOUS AND CODEBOOK OBLIVIOUS G-IC

The only requirement for the implementation of the TINnoTS inner bound in (5) is to have symbol synchronization and knowledge of the channel gains at all the terminals. Therefore, our

TINnoTS achievable strategy applies to a large class of channels, besides the model considered thus far. Next, we outline two such examples for which very little was known in the past.

The first example is the block asynchronous G-IC, which is information unstable [15] and thus no single-letter capacity expression can be derived for it. Nonetheless, we are able to show that the capacity of this channel is to within a gap of the capacity of the fully synchronized channel. The second example is the G-IC with partial codebook knowledge at both receivers [40], which prevents using joint decoding or successive interference cancellation at the decoders. Still, we are able to show that the capacity of this channel is to within a gap of the capacity of the channel with full codebook knowledge.

The applications to oblivious and asynchronous ICs somewhat surprisingly implies that much less “global coordination” between nodes is needed than one might initially expect: synchronism and codebook knowledge might not be critical if one is happy with ‘approximate’ capacity results.

A. Block Asynchronous G-IC

Consider a G-IC with the following input-output relationship

$$Y_{1,t} = h_{11}X_{1,t} + h_{12}X_{2,t-D_1} + Z_{1,t}, \quad (103a)$$

$$Y_{2,t} = h_{21}X_{1,t-D_2} + h_{22}X_{2,t} + Z_{2,t}, \quad (103b)$$

for $t \in \mathbb{Z}^+$, and $X_{i,j}$ user i 's input to the channel at channel use j , $X_{i,j} = 0$ for $j < 0$ (similarly for $Y_{i,j}$ and $Z_{i,j}$), where the delay D_i , $i \in [1 : 2]$, is chosen at the beginning of the transmission and held fixed thereafter. The channel is termed totally asynchronous if delay D_i is uniform on all n [15]. Except for the introduction of random delay all definitions are identical to those given in Section I. In [15] it has been shown that $\mathcal{R}_{\text{in}}^{\text{TINnoTS}}$ in (5) is achievable for the channel in (103). Moreover, because lack of synchronization can only harm communications, the outer bound in Proposition 6 is a valid outer bound for the asynchronous G-IC. Therefore, all of our previous results hold and we have:

Lemma 10. *For the block asynchronous G-IC the TINnoTS achievable region is to within an additive gap of the capacity of the fully synchronized G-IC, where the gap is given in Theorems 7 and 8.*

B. IC with No Codebook Knowledge

IC with partial codebook knowledge, or oblivious receivers (IC-OR), has been introduced in [40]. This channel model is practically relevant because it models the inability to use sophisticated decoding techniques such as joint decoding or successive inference cancellation. Recently, in [8], for the IC-OR with partial codebook knowledge at one receiver, it has been shown that using Gaussian input at the transmitter corresponding to the oblivious receiver and a mixed input at the transmitter corresponding to non-oblivious receiver is to within a constant gap from the capacity of the classical G-IC with full codebook knowledge. In [40] it was shown that for IC-OR with both oblivious receivers the capacity is given by

$$\mathcal{C}^{\text{IC-OR}} = \bigcup_{P_Q P_{X_1|Q} P_{X_2|Q}} \left\{ \begin{array}{l} R_1 \leq I(X_1; Y_1|Q) \\ R_2 \leq I(X_2; Y_2|Q) \end{array} \right\}. \quad (104)$$

Note that the region in (104) is very similar to TINoTS region in (4) and $\mathcal{C}^{\text{IC-OR}}$ is upper bounded by the classical G-IC outer bound in Proposition 6. The set of optimizing distributions for (104) and the cardinality bound for the alphabet of Q are not known [40, Section III.A]. Based on our previous results, we have that:

Lemma 11. *For the G-IC with partial codebook knowledge the TINnoTS achievable region is to within an additive gap of the capacity of the G-IC with full codebook knowledge, where the gap is given in Theorems 7 and 8.*

VIII. TINNOTS WITH MIXED INPUTS IN PRACTICE

A. A Simple TINnoTS Receiver in Very Strong Interference

In the Introduction we mentioned that the optimal MAP decoder in an additive non-Gaussian noise channel, which one could implement for TIN when treating a non-Gaussian interference as noise, could be very complex. In the following we give an example of an approximate MAP decoder that is very simple to implement, thus making TINnoTS competitive in practical applications.

Let X_1, X_2 be from the PAM(N, d) with $N = 2Q + 1, Q \in \mathbb{N}$, and $d^2 = \frac{12}{N^2-1} = \frac{3}{Q(Q+1)}$. The restrictions to an odd number of points is just for simplicity of writing the constellation points. The received signal is

$$Y = \left(\sqrt{S} n_1 + \sqrt{I} n_2 \right) d + Z_G, \quad Z_G \sim \mathcal{N}(0, 1),$$

for some $(n_1, n_2) \in [-Q : Q]^2$ chosen independently with uniform probability. The condition in (21b) is verified when

$$(2Q + 1)^2 S \leq 1, \quad (105)$$

which corresponds to the very strong interference regime. In the regime identified by (105), i.e., where the received points do not ‘overlap’ as in Fig 2, the decoder could simply “modulo-out” the interference by “folding” the signal Y onto the interval $\mathcal{I} := [-\sqrt{1}d/2, +\sqrt{1}d/2]$. By doing so the resulting signal, given by

$$Y' = \left[\sqrt{S} n_1 d + Z' \right]_{\text{mod } \mathcal{I}}, \quad Z' := [Z_G]_{\text{mod } \mathcal{I}},$$

would be interference-free. Since

$$\begin{aligned} \Pr[Y' \neq \sqrt{S} n_1 d + Z_G] &\leq \Pr[|Z_G| \geq \sqrt{1}d/2 - \sqrt{S}Qd] \\ &\stackrel{\text{from (105)}}{\leq} \Pr[|Z_G| \geq \sqrt{S}/2], \end{aligned}$$

and since $\Pr[|Z_G| \geq \sqrt{S}/2]$ is also an upper bound to the probability of error for PAM input on a Gaussian channel, we see that the simple modulo operation at the receiver results in a symbol-error rate that is at most double that of an interference-free Gaussian channel with the same PAM input.

B. Actual vs. Analytic Gap

Here we compare the gap derived in Theorems 7 and 8 to the actual gap evaluated numerically. The point is to show that our analytical closed-form (worst case scenario) bounds can be quite conservative and thus underestimate the actual achievable rates.

For example, we showed that in the very strong interference regime the TINnoTS achievable region with discrete inputs is at most $\frac{1}{2} \log\left(\frac{2\pi e}{3}\right)$ bits from capacity; the capacity in this case is the same as two parallel interference-free links. Consider the symmetric G-IC in very strong interference and the symmetric rate $R_1 = R_2 = R_{\text{sym}}(S)$ with the same PAM input for each user, where the number of points is chosen as in (38a). Fig. 4a shows $\text{gap}(S) := I_g(S) - R_{\text{sym}}(S)$ vs. S expressed in dB, where

- The red line is the theoretical gap from Theorem 7, approximately $\frac{1}{2} \log\left(\frac{2\pi e}{3}\right) = 1.25$ bits;
- The green line is the gap by lower bounding $R_{\text{sym}}(S)$ with the Ozarow-Wyner-B bound in Proposition 1, where the minimum distance of the received constellation was computed

exactly (rather than lower bounded by Proposition 2); the gap in this case is approximately 0.75 bits;

- The magenta line is the gap by lower bounding $R_{\text{sym}}(S)$ by the ‘full DTD-ITA’14 bound’ in (18a), the gap in this case is approximately 0.37 bits; and
- The cyan line is the gap when $R_{\text{sym}}(S)$ is evaluated by Monte Carlo simulation; The gap in this case tends to the ultimate “shaping loss” $\frac{1}{2} \log\left(\frac{\pi e}{6}\right) = 0.25$ bits at large S ; this shows that the actual gap is about 1 bit lower than the theoretical gap.

The figure also shows that the lower bound in (18a) actually gives the tightest lower bound for the mutual information, but it is unfortunately not easy to deal with analytically.

We next consider the symmetric G-IC in strong interference. Theorem 7 upper bounds the gap in this regime by $\text{gap}(S) \leq \frac{1}{2} \log\left(\frac{2\pi e}{3}\right) + \frac{1}{2} \log\left(1 + 8 \frac{(1+1/2 \ln(1+S))^2}{\gamma^2}\right)$ where $\gamma \in (0, 1]$ is the measure of the outage set (i.e., those channel gains for which the gap lower bound is not valid). If we were to make the measure of the outage set very small, then we could end up finding that the gap is actually larger than capacity. Consider the case $S = 30$ dB and $I = S^{1.49} = 44.7$ dB; with $\gamma = 0.1$ it easy to see that $\frac{1}{2} \log\left(\frac{2\pi e}{3}\right) + \frac{1}{2} \log\left(1 + 8 \frac{(1+1/2 \ln(1+S))^2}{\gamma^2}\right) = 6.977$ bits, which is larger than the interference-free capacity $I_g(S) = 4.9836$ bits. This implies that our bounding steps, done for the sake of analytical tractability and especially meaningful at high SNR, are too crude for this specific example (where our result states the trivial fact that zero rate for each user is achievable to within $I_g(S)$ bits). We aim to convey next that, despite the fact that the closed-form gap result underestimates the achievable rates, it nonetheless provides valuable insights into the performance of practical systems, that is, that TINnoTS with discrete inputs performs quite well in the strong interference regime (where capacity is achieved by Gaussian codebooks and joint decoding of interfering and intended messages). To this end, Fig. 4b shows the achievable rate region for the symmetric G-IC with $S = 30$ dB and $I = S^{1.49} = 44.7$ dB and where the users employ a PAM input with the number of points given by (49). We observe

- The navy blue line shows the pentagon-shaped capacity region in (43);
- The red point at the origin is the lower bound on the achievable rates from Theorem 7 with $\gamma = 0.1$;
- The green line is the achievable region when the rates are lower bounded by the Ozarow-Wyner-B bound in Proposition 1, where the minimum distances of the received constellations were computed exactly (rather than lower bounded by Proposition 3);

- For the magenta line we used the DTD-ITA'14-A lower bound in (18a); and
- For the cyan line we evaluated the rates by Monte Carlo simulation.

The reason why the green region has so many ‘ups and downs’ is because the Ozarow-Wyner-B bound in Proposition 1 depends on the constellation through its minimum distance; as we already saw in Fig. 3, the minimum distance is very sensitive to the fractional values of the channel gains, which makes the corresponding bound looks very irregular. On the other hand, the magenta region is based on the lower bound in (18a), which depends on the whole distance spectrum of the received constellation and as a consequence the corresponding bound looks smoother. The cyan region is the smoothest of all; its largest gap occurs at the symmetric rate point and is less than 0.7 bits – as opposed to the theoretical gap of 4.9836 bits. We thus conclude that, despite the large theoretical gap, a PAM input is quite competitive in this example.

C. Mixed (Gaussian+Discrete) vs. Discrete (Discrete+Discrete) Inputs

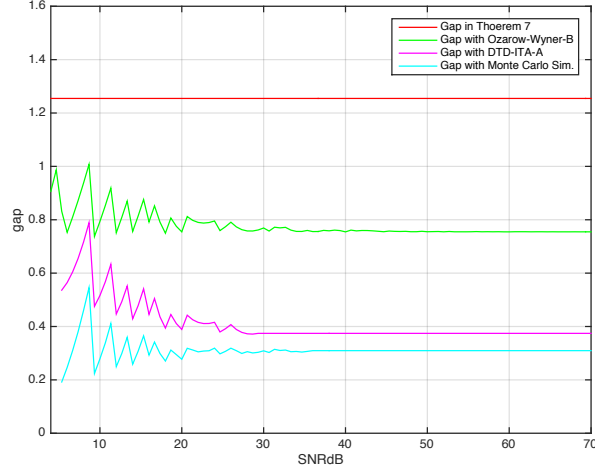
In the previous Sections we showed that TINnoTS with mixed (Gaussian+Discrete) inputs achieves the capacity to within a gap for several channels of interest. Practically, it may be interesting to understand what performance can be guaranteed when inputs are fully discrete, i.e., they do not contain a Gaussian component.

For the symmetric G-IC the following can be shown. Consider the TINnoTS region with $X_u \sim \text{PAM}(N_u, d_u)$ such that the power constraints are met, that is, $\frac{N_u^2 - 1}{12} d_u^2 \leq 1$ for all $u \in [1 : 2]$, and lower bound the mutual informations with Proposition 1. Then, TINnoTS achieves the outer bound in Proposition 6 in very weak and in strong interference only, that is, for those regimes where ‘rate splitting’ was not used in Theorem 7. The proof of this result is omitted for sake of space. Thus it appears that in the moderately weak interference regime mixed inputs composed of ‘two-layers’ are necessary.

The next question we ask is thus whether we can show the same gap result of Theorem 7 for the moderately weak interference regime by using inputs that are the superposition of two PAM constellations, rather than a PAM and a Gaussian. The next proposition shows that the answer is in the affirmative, i.e., it is possible to ‘switch’ between Gaussian+Discrete and Discrete+Discrete inputs up to an additive gap.

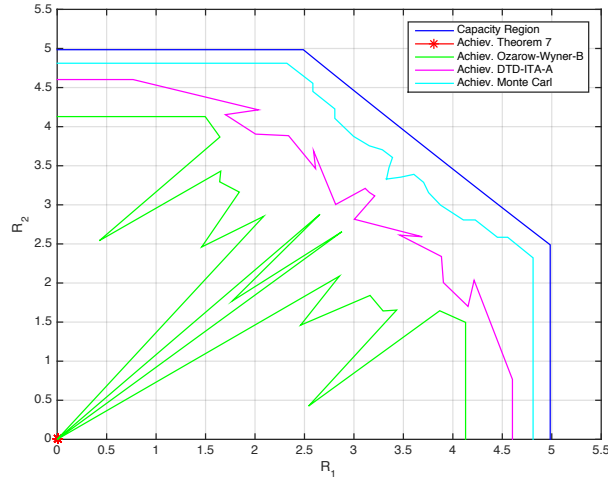
Proposition 12. *Let*

$$X_D := X_c + X_p,$$



(a) Gap in the very strong interference regime vs. S for

$$I = (S(1+S))^{1.2} \approx S^{2.4}.$$



(b) Rate region in the strong interference regime for

$$S = 30 \text{ dB and } I = S^{1.49} = 44.7 \text{ dB.}$$

Fig. 4: Comparing Analytic with Numerical Gaps.

where $X_c \sim \text{discrete} : d_{\min}(X_c) > 0$,

$X_p \sim \text{discrete} : d_{\min}(X_p) > 0$,

$$X_M := X_c + X_g,$$

where $X_g \sim \mathcal{N}(0, \mathbb{E}[|X_g|^2])$ such that $\mathbb{E}[|X_p|^2] = \mathbb{E}[|X_g|^2]$,

where X_c, X_g and X_p are mutually independent. Then, for $Z_G \sim \mathcal{N}(0, 1)$ independent of

everything else, we have

$$\begin{aligned}
I(X_D; gX_D + Z_G) - I(X_M; gX_M + Z_G) &\leq \frac{1}{2} \log(2), \\
I(X_M; gX_M + Z_G) - I(X_D; gX_D + Z_G) &\leq \frac{1}{2} \log\left(\frac{\pi e}{3}\right) \\
&\quad + \frac{1}{2} \log\left(1 + \frac{12}{g^2 d_{\min(X_D)}^2}\right).
\end{aligned}$$

Proof: The first inequality follows since

$$\begin{aligned}
&I(X_D; gX_D + Z_G) \\
&= I(X_c, X_p; gX_c + gX_p + Z_G) \\
&= I(X_p; gX_c + gX_p + Z_G) + I(X_c; gX_c + gX_p + Z_G | X_p) \\
&= I(X_p; gX_p + N) |_{N:=gX_c+Z_G} + I(X_c; gX_c + Z_G) \\
&\stackrel{(a)}{\leq} I(X_g; gX_g + N) |_{N:=gX_c+Z_G} \\
&\quad + \frac{1}{2} \log(2) + I(X_c; gX_c + Z_G) \\
&= I(X_M; gX_M + Z) + \frac{1}{2} \log(2),
\end{aligned}$$

where in (a) we used [12, Theorem 1], which states that a Gaussian input for non-Gaussian additive noise channel results in at most 1/2 bit loss.

The second inequality follows since

$$\begin{aligned}
I(X_M; gX_M + Z_G) &\leq \mathsf{I}_g(g^2 \text{Var}[X_M]) = \mathsf{I}_g(g^2 \text{Var}[X_D]) \\
&\stackrel{(b)}{\leq} I(X_D; gX_D + Z_G) + \frac{1}{2} \log\left(\frac{\pi e}{3}\right) \\
&\quad + \frac{1}{2} \log\left(1 + \frac{12}{g^2 d_{\min(X_D)}^2}\right),
\end{aligned}$$

where in (b) we used the bound in Proposition 1. ■

The question left is thus why ‘two-layer’ inputs, i.e., that comprise two random variables, are needed for approximate optimality in the moderately weak interference regime. Although at this point we do not have an answer for this question, the intuition for the moderately weak interference regime is as follows. With ‘single-layer’ PAM inputs and for the given power constraints, the number of points needed to attain a desired rate pair on the convex closure of the outer bound result in a minimum distance at the receivers that is too small. It may be

that with ‘two-layer’ PAM inputs one effectively soft-estimates one of the layers whose effect can thus be removed from the received signal, thereby behaving as if there was an interfering common message jointly decoded at the non-intended receiver. Further investigation is needed to understand whether ‘multi-layer’ inputs are indeed necessary.

IX. CONCLUSION

We evaluated a very simple, generally applicable lower bound, that neither requires joint decoding nor block synchronization, to the capacity of the Gaussian interference channel. This treating-interference-as-noise lower bound without time-sharing was evaluated for inputs that are a mixture of discrete and Gaussian random variables. We showed that, through careful choice of the mixed input parameters, namely the number of points of the discrete part and the amount of power assigned to the Gaussian part (that in general depends on the channel gains and on which point on the convex closure of the outer bound one wants to attain) the capacity of the classical Gaussian interference channel can be attained to within a gap. This result is of interest in several channels where this lower bound applies, such as block asynchronous channels and channels with partial codebook knowledge.

The techniques inspired by this paper have been recently used in [41] to find approximated capacity result for a problem of communication with a MMSE disturbance constraint. In another line of work in [42] using the links between estimation theory and information theory the Ozarow-Wyner bound has been sharpened. Consequently, this improves all of the gap results in this work. Extension to other channel models and to more than two users are the subject of current investigation.

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APPENDIX A
PROOF OF (18c)

To prove the lower bound in (18a) we first find a lower bound on the differential entropy of $Y = X_D + Z_G$. To that end let $p_i := \mathbb{P}[X_D = s_i], i \in [1 : N]$, then Y has the following Gaussian mixture density

$$Y \sim P_Y(y) := \sum_{i \in [1:N]} p_i \mathcal{N}(y; s_i, 1). \quad (106)$$

where

$$\mathcal{N}(x; \mu, \sigma^2) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}.$$

We have

$$\begin{aligned} & -h(Y) \\ &= \int P_Y(y) \log(P_Y(y)) dy \\ &\stackrel{(a)}{\leq} \log \int P_Y(y) P_Y(y) dy \\ &= \log \int \left(\sum_{i \in [1:N]} p_i \mathcal{N}(y; s_i, 1) \right)^2 dy \\ &= \log \left(\sum_{(i,j) \in [1:N]^2} p_i p_j \int \mathcal{N}(y; s_i, 1) \mathcal{N}(y; s_j, 1) dy \right) \\ &= \log \left(\sum_{(i,j) \in [1:N]^2} \frac{p_i p_j}{\sqrt{4\pi}} e^{-\frac{(s_i - s_j)^2}{4}} \right. \\ &\quad \cdot \left. \int \mathcal{N}\left(y; \frac{s_i + s_j}{2}, \frac{1}{2}\right) dy \right) \\ &\stackrel{(b)}{=} \log \left(\sum_{(i,j) \in [1:N]^2} \frac{p_i p_j}{\sqrt{4\pi}} e^{-\frac{(s_i - s_j)^2}{4}} \right) \\ &\stackrel{(c)}{\leq} \log \left(\sum_{i \in [1:N]} \frac{p_i^2}{\sqrt{4\pi}} + \sum_{i \in [1:N]} \frac{p_i(1 - p_i)}{\sqrt{4\pi}} e^{-\frac{d_{\min}^2(X_D)}{4}} \right), \end{aligned}$$

which for uniformly distributed inputs simplifies to

$$I(X_D; X_D + Z_G) = h(X_D + Z_G) - h(Z_G) \geq \log(N) - \mathbf{gap}_{(108)}, \quad (107)$$

$$\text{gap}_{(108)} := \frac{1}{2} \log \left(\frac{e}{2} \right) + \log \left(1 + (N-1)e^{-\frac{d_{\min}^2(X_D)}{4}} \right), \quad (108)$$

where the (in)equalities follow from: (a) Jensen's inequality; (b) $\int \mathcal{N}(y; \mu, \sigma^2) dy = 1$; and (c) $d_{\min}(X_D) \leq |s_i - s_j|, \forall i \neq j$. Combining this bound with the fact that mutual information is non-negative proves the claimed lower bound.

APPENDIX B

PROOF OF PROPOSITION 3

For convenience let $\mathcal{S} := \text{supp}(h_{xx}X + h_{yy}Y)$. To prove that $|S| = |X||Y|$ a.e. we look at the measure of the set such that $|S| \neq |X||Y|$, that is, a set for which there exists $s_i = h_{xx}x_i + h_{yy}y_i$ and $s_j = h_{xx}x_j + h_{yy}y_j$ such that $s_i = s_j$ for some $i \neq j$; hence, we are interested in characterizing the measure of the set

$$A := \left\{ (h_{xx}, h_{yy}) \in [0, 1]^2 : \begin{array}{l} h_{xx}x_i + h_{yy}y_i = h_{xx}x_j + h_{yy}y_j \\ (x_i, y_i) \neq (x_j, y_j) \\ \forall x_i, x_j \in X \text{ and } \forall y_i, y_j \in Y \end{array} \right\}. \quad (109)$$

Define

$$A(i, j) := \left\{ (h_{xx}, h_{yy}) \in [0, 1]^2 : \begin{array}{l} h_{xx}x_i + h_{yy}y_i = h_{xx}x_j + h_{yy}y_j, \\ \text{s.t. } (x_i, y_i) \neq (x_j, y_j) \end{array} \right\}. \quad (110)$$

By the sub-additivity of measure we have

$$m(A) = m \left(\bigcup_{i,j} A(i, j) \right) \leq \sum_{i,j} m(A(i, j)). \quad (111)$$

For fixed x_i, x_j, y_i, y_j the set $A(i, j)$ is a line in $(h_{xx}, h_{yy}) \in \mathbb{R}^2$ and hence

$$m(A(i, j)) = 0.$$

Thus, in (111) we have a countable sum of sets of measure zero, which implies that $m(A) = 0$.

Next, we bound the minimum distance

$$d_{\min}(\mathcal{S}) := \min_{i \neq j} \{|s_i - s_j| : s_i, s_j \in \mathcal{S}\},$$

with $|s_i - s_j| = |h_{xx}x_i + h_{yy}y_i - h_{xx}x_j - h_{yy}y_j|$. We distinguish two cases:

Case 1) $x_i = x_j$ and $y_i \neq y_j$, or $x_i \neq x_j$ and $y_i = y_j$: then trivially

$$|s_i - s_j| \geq |h_{yy}|d_{\min(Y)}, \text{ or}$$

$$|s_i - s_j| \geq |h_{xx}|d_{\min(X)}.$$

Case 2) $x_i \neq x_j$ and $y_i \neq y_j$: Let $z_* \in \mathbb{Z}$, then

$$\begin{aligned} & |s_i - s_j| \\ &= |h_{xx}x_i + h_{yy}y_i - h_{xx}x_j - h_{yy}y_j| \\ &= |h_{xx}(x_i - x_j) - h_{yy}(y_j - y_i)| \\ &= |h_{xx}d_{\min(X)}(z_{xi} - z_{xj}) - h_{yy}d_{\min(Y)}(z_{yj} - z_{yi})| \\ &= |a_{xx}\bar{a}z_x - b_{yy}\bar{b}z_y| \end{aligned}$$

where $z_x = (z_{xi} - z_{xj})$ and $z_y = (z_{yj} - z_{yi})$ and $a_{xx}\bar{a} = h_{xx}d_{\min(X)}$, $b_{yy}\bar{b} = h_{yy}d_{\min(Y)}$.

That is a_{xx}, b_{yy} are the fractional parts of and \bar{a} and \bar{b} are the integer parts $h_{xx}d_{\min(X)}$ and $h_{yy}d_{\min(Y)}$ respectively. Hence, by Lemma 13 in Appendix C we have that

$$\begin{aligned} & |s_i - s_j| \\ &\geq \gamma \max \left(\frac{|h_{xx}|d_{\min(X)}}{2|Y|(1 + \log(|X|))}, \frac{|h_{yy}|d_{\min(Y)}}{2|X|(1 + \log(|Y|))} \right) \\ &\geq \kappa_{\gamma, |X|, |Y|} \max \left(\frac{|h_{xx}|d_{\min(X)}}{|Y|}, \frac{|h_{yy}|d_{\min(Y)}}{|X|} \right) \end{aligned}$$

up to an outage set of measure γ where $\kappa_{\gamma, |X|, |Y|} := \frac{\gamma}{1 + \ln(\max(|X|, |Y|))}$ and $\gamma \in (0, 1]$. Next, by taking the minimum over both cases we arrive at the result in Proposition 3.

APPENDIX C

MINIMUM DISTANCE AUXILIARY LEMMA

Lemma 13. *Let $(a_{xx}, b_{yy}) \in [0, 1]^2$ then for fixed integers $\bar{a}, \bar{b} \in \mathbb{Z}$. Then the function*

$$f(a_{xx}, b_{yy}) = \min_{z_x, z_y} |a_{xx}\bar{a}z_x - b_{yy}\bar{b}z_y|,$$

subject to the constraints

$$z_x \in [-N_x : N_x] \setminus \{0\},$$

$$z_y \in [-N_y : N_y] \setminus \{0\},$$

satisfies

$$f(a_{xx}, b_{yy}) \geq \frac{\gamma}{2} \max \left(\frac{b_{yy} \bar{b}}{N_x(1 + \ln(N_y))}, \frac{a_{xx} \bar{a}}{N_y(1 + \ln(N_x))} \right),$$

for all $(a_{xx}, b_{yy}) \in [0, 1]^2$ except for an outage set of Lebesgue measure γ for any $\gamma \in (0, 1]$.

Proof: First observe that w.l.o.g. we can assume that $a_{xx} \bar{a}, b_{yy} \bar{b} \in \mathbb{R}^+$ and $z_x \in [1 : N_x]$ and $z_y \in [1 : N_y]$. This is because if $\text{sign}(a_{xx} \bar{a} z_x) \neq \text{sign}(b_{yy} \bar{b} z_y)$ then the function is minimized by $|z_x| = 1$ and $|z_y| = 1$ and attains a value of $f = |a_{xx} \bar{a}| + |b_{yy} \bar{b}|$. Furthermore, we let

$$\begin{aligned} A_\epsilon &= \left\{ (a_{xx}, b_{yy}) \in [0, 1]^2 : \min_{\substack{1 \leq z_x \leq N_x, \\ 1 \leq z_y \leq N_y}} |a_{xx} \bar{a} z_x - b_{yy} \bar{b} z_y| > \epsilon \right\} \\ &= \bigcap_{\substack{1 \leq z_x \leq N_x, \\ 1 \leq z_y \leq N_y}} \{ (a_{xx}, b_{yy}) \in [0, 1]^2 : |a_{xx} \bar{a} z_x - b_{yy} \bar{b} z_y| > \epsilon \} \\ &= \bigcap_{\substack{1 \leq z_x \leq N_x, \\ 1 \leq z_y \leq N_y}} A_\epsilon(z_x, z_y), \end{aligned}$$

where

$$A_\epsilon(z_x, z_y) = \{ (a_{xx}, b_{yy}) \in [0, 1]^2 : |a_{xx} \bar{a} z_x - b_{yy} \bar{b} z_y| > \epsilon \}.$$

The shape of $A_\epsilon(z_x, z_y)$ is shown on Fig. 5. Let A_ϵ^c be the complement of A_ϵ where we have

$$A_\epsilon^c = \bigcup_{1 \leq z_x \leq N_x, 1 \leq z_y \leq N_y} A_\epsilon^c(z_x, z_y) \quad (112)$$

where

$$A_\epsilon^c(z_x, z_y) = \{ (a_{xx}, b_{yy}) \in [0, 1]^2 : |a_{xx} \bar{a} z_x - b_{yy} \bar{b} z_y| \leq \epsilon \}. \quad (113)$$

Next, we find the measure of the set A_ϵ^c as follows:

$$\begin{aligned} m(A_\epsilon^c) &= m \left(\bigcup_{1 \leq z_x \leq N_x, 1 \leq z_y \leq N_y} A_\epsilon^c(z_x, z_y) \right) \\ &\leq \sum_{1 \leq z_x \leq N_x, 1 \leq z_y \leq N_y} m(A_\epsilon^c(z_x, z_y)), \end{aligned}$$

where the last inequality is due to the sub-additive of measure.

Next, we compute $m(A_\epsilon^c(z_x, z_y))$ in (114).

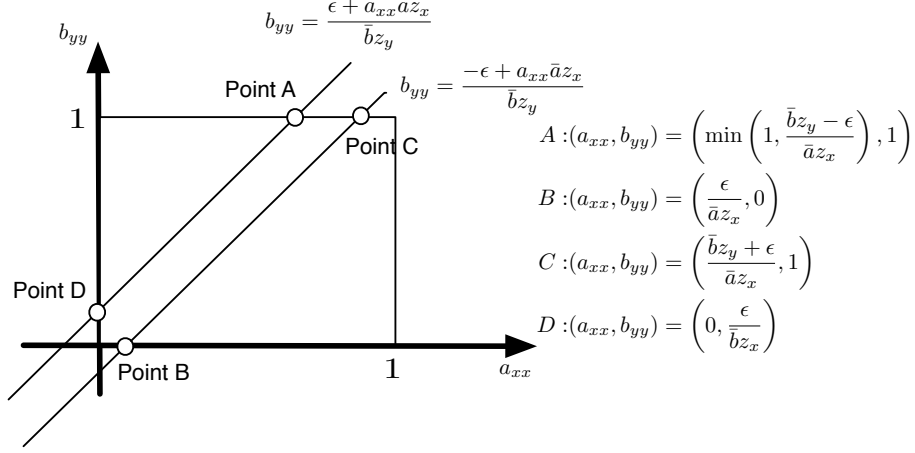


Fig. 5: Shape of the outage strip.

$$\begin{aligned}
m(A_\epsilon^c(z_x, z_y)) &= \int_{a_{xx}=0}^{a_{xx}=\text{Point A}} \frac{\epsilon + a_{xx} \bar{a} z_x}{\bar{b} z_y} da_{xx} - \int_{a_{xx}=\text{Point B}}^{a_{xx}=\text{Point A}} \frac{-\epsilon + a_{xx} \bar{a} z_x}{\bar{b} z_y} da_{xx} \\
&= \int_{a_{xx}=0}^{a_{xx}=\min\left(1, \frac{\bar{b} z_y - \epsilon}{\bar{a} z_x}\right)} \frac{\epsilon + a_{xx} \bar{a} z_x}{\bar{b} z_y} da_{xx} - \int_{a_{xx}=\frac{\epsilon}{\bar{a} z_x}}^{a_{xx}=\min\left(1, \frac{\bar{b} z_y - \epsilon}{\bar{a} z_x}\right)} \frac{-\epsilon + a_{xx} \bar{a} z_x}{\bar{b} z_y} da_{xx} \\
&= \frac{a_{xx} (2\epsilon + \bar{a} a_{xx} z_x)}{2 \bar{b} z_y} \Big|_{a_{xx}=0}^{a_{xx}=\min\left(1, \frac{\bar{b} z_y - \epsilon}{\bar{a} z_x}\right)} - \left(-\frac{a_{xx} (2\epsilon - \bar{a} a_{xx} z_x)}{2 \bar{b} z_y} \Big|_{a_{xx}=\frac{\epsilon}{\bar{a} z_x}}^{a_{xx}=\min\left(1, \frac{\bar{b} z_y - \epsilon}{\bar{a} z_x}\right)} \right) \\
&= \frac{\min\left(1, -\frac{(\epsilon - \bar{b} z_y)}{\bar{a} z_x}\right) \left(2\epsilon + \bar{a} z_x \min\left(1, -\frac{(\epsilon - \bar{b} z_y)}{\bar{a} z_x}\right)\right)}{2 \bar{b} z_y} - \frac{\left(\epsilon - \bar{a} z_x \min\left(1, -\frac{(\epsilon - \bar{b} z_y)}{\bar{a} z_x}\right)\right)^2}{2 \bar{a} \bar{b} z_x z_y} \\
&= -\frac{\epsilon \left(\epsilon - 4 \bar{a} z_x \min\left(1, -\frac{(\epsilon - \bar{b} z_y)}{\bar{a} z_x}\right)\right)}{2 \bar{a} \bar{b} z_x z_y}.
\end{aligned} \tag{114}$$

Next, compute $m(A_\epsilon^c)$ as follows

$$\begin{aligned}
m(A_\epsilon^c) &= \sum_{1 \leq z_x \leq N_x, 1 \leq z_y \leq N_y} -\frac{\epsilon \left(\epsilon - 4 \bar{a} z_x \min\left(1, -\frac{(\epsilon - \bar{b} z_y)}{\bar{a} z_x}\right)\right)}{2 \bar{a} \bar{b} z_x z_y} \\
&= \sum_{1 \leq z_x \leq N_x, 1 \leq z_y \leq N_y} \frac{-\epsilon^2}{2 \bar{a} \bar{b} z_x z_y}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{1 \leq z_x \leq N_x, 1 \leq z_y \leq N_y} \frac{4\epsilon \bar{a} z_x \min\left(1, -\frac{(\epsilon - \bar{b} z_y)}{\bar{a} z_x}\right)}{2 \bar{a} \bar{b} z_x z_y} \\
& \leq \sum_{1 \leq z_x \leq N_x, 1 \leq z_y \leq N_y} \frac{4\epsilon \bar{a} z_x \min\left(1, -\frac{(\epsilon - \bar{b} z_y)}{\bar{a} z_x}\right)}{2 \bar{a} \bar{b} z_x z_y}.
\end{aligned}$$

The term $\min\left(1, -\frac{(\epsilon - \bar{b} z_y)}{\bar{a} z_x}\right)$ can be upper bounded in two different ways

$$\min\left(1, -\frac{(\epsilon - \bar{b} z_y)}{\bar{a} z_x}\right) \leq 1, \quad (115)$$

$$\min\left(1, -\frac{(\epsilon - \bar{b} z_y)}{\bar{a} z_x}\right) \leq -\frac{(\epsilon - \bar{b} z_y)}{\bar{a} z_x}. \quad (116)$$

With the first upper bound in (115) we get

$$\begin{aligned}
m(A_\epsilon^c) & \leq \sum_{1 \leq z_x \leq N_x, 1 \leq z_y \leq N_y} \frac{4\epsilon \bar{a} z_x}{2 \bar{a} \bar{b} z_x z_y} \\
& \leq \frac{2\epsilon N_x (1 + \ln(N_y))}{\bar{b}},
\end{aligned} \quad (117)$$

where for the last inequality we have used $\sum_{z_y=1}^{N_y} \frac{1}{z_y} \leq 1 + \ln(N_y)$. With the second upper bound in (116) we get

$$\begin{aligned}
m(A_\epsilon^c) & \leq \sum_{1 \leq z_x \leq N_x, 1 \leq z_y \leq N_y} \frac{4\epsilon (\bar{b} z_y - \epsilon)}{2 \bar{a} \bar{b} z_x z_y} \\
& \leq \sum_{1 \leq z_x \leq N_x, 1 \leq z_y \leq N_y} \frac{4\epsilon}{2 \bar{a} z_x} \\
& \leq \frac{2\epsilon N_y (1 + \ln(N_x))}{\bar{a}}.
\end{aligned} \quad (118)$$

So by taking the tightest of the two bounds in (117) and in (118) we get

$$m(A_\epsilon^c) \leq \min\left(\frac{2\epsilon N_x (1 + \ln(N_y))}{\bar{b}}, \frac{2\epsilon N_y (1 + \ln(N_x))}{\bar{a}}\right).$$

Now let $m(A_\epsilon^c) = \gamma$ for some $\gamma \in [0, 1]$ then we have that

$$\gamma \leq \epsilon \min\left(\frac{2N_x (1 + \ln(N_y))}{\bar{b}}, \frac{2N_y (1 + \ln(N_x))}{\bar{a}}\right).$$

Next, by solving for ϵ in terms of measure of the outage,

$$\epsilon \geq \frac{\gamma}{\min\left(\frac{2N_x (1 + \ln(N_y))}{\bar{b}}, \frac{2N_y (1 + \ln(N_x))}{\bar{a}}\right)}$$

$$\begin{aligned}
&= \gamma \max \left(\frac{\bar{b}}{2N_x(1 + \ln(N_y))}, \frac{\bar{a}}{2N_y(1 + \ln(N_x))} \right) \\
&\geq \gamma \max \left(\frac{b_{yy}\bar{b}}{2N_x(1 + \ln(N_y))}, \frac{a_{xx}\bar{a}}{2N_y(1 + \ln(N_x))} \right).
\end{aligned}$$

This concludes the proof. ■

APPENDIX D

GAP FOR $l \leq S \leq 1 + l$

Outer Bound for $l \leq S \leq 1 + l$: It is well known that when $S \approx l$ time-division is approximately optimal [23]. In this regime we outer bound the capacity region by the sum-rate constraint in (36c) only, which in the symmetric case is

$$\begin{aligned}
R_1 + R_2 &\leq I_g(S) - I_g(l) + I_g(S + l) \\
&= I_g(S) + I_g\left(\frac{S}{1 + l}\right) \\
&\leq I_g(S) + \frac{1}{2} \log(2),
\end{aligned}$$

that is

$$\mathcal{R}_{\text{out}}^{(D)} = \bigcup_{t \in [0,1]} \left\{ \begin{array}{l} R_1 \leq t (I_g(S) + \frac{1}{2} \log(2)) \\ R_2 \leq (1 - t) (I_g(S) + \frac{1}{2} \log(2)) \end{array} \right\}.$$

Inner Bound for $l \leq S \leq 1 + l$: We only use the discrete part of the mixed inputs and set

$$N_1 = N_d(S_{1,t}), \quad S_{1,t} := (1 + S)^t - 1 \leq S, \quad (119a)$$

$$N_2 = N_d(S_{2,t}), \quad S_{2,t} := (1 + S)^{1-t} - 1 \leq S, \quad (119b)$$

$$\delta_1 = 0, \quad (119c)$$

$$\delta_2 = 0. \quad (119d)$$

Note that

$$N_1^2 N_2^2 - 1 \leq (1 + S_{1,t})(1 + S_{2,t}) - 1 = S. \quad (119e)$$

We lower bound the minimum distance of the sum-set constellations as in (47) and we get

$$\min_{i \in [1:2]} \frac{d_{\min}^2(S_i)}{12 \kappa_{\gamma, N_1, N_2}^2} \geq \min \left(\frac{\min(S, l)}{\max(N_1^2, N_2^2) - 1}, \frac{\max(S, l)}{N_1^2 N_2^2 - 1} \right)$$

$$\begin{aligned}
& \text{for } l \leq S \text{ and (119)} \\
& \geq \min \left(\frac{l}{S}, \frac{S}{S} \right) \\
& \text{for } S \leq 1+l \\
& \geq \min \left(\frac{l}{1+l}, 1 \right) \\
& = \frac{l}{1+l} \\
& \text{for } 1 \leq l \leq \frac{1}{2} \\
& \geq \frac{1}{2}.
\end{aligned}$$

Gap for $l \leq S \leq 1+l$: Similarly to the strong interference regime, we can upper bound the difference between the upper and lower bounds as

$$\begin{aligned}
\text{gap} & \leq \max \left(I_g(S_{1,t}) + \frac{t}{2} \log(2) - \log(N_d(S_{1,t})), \right. \\
& \quad \left. I_g(S_{2,t}) + \frac{1-t}{2} \log(2) - \log(N_d(S_{2,t})) \right) \\
& \quad + \frac{1}{2} \log \left(\frac{2\pi e}{12} \right) + \frac{1}{2} \log \left(1 + \frac{2}{\kappa_{\gamma, N_1, N_2}^2} \right) \\
& \leq \text{Gap}(4, 8).
\end{aligned}$$

APPENDIX E

AUXILIARY RESULTS FOR REGIME WEAK1

We derive here some auxiliary results for the regime in (55), namely

$$\begin{aligned}
(1+l) & \leq S \leq l(1+l), \\
\frac{1+S}{1+l+\frac{S}{1+l}} & \leq \frac{1+l+\frac{S}{1+l}}{1+\frac{S}{1+l}}.
\end{aligned}$$

A. Derivation of (64)

The parameters of the mixed inputs are given in (63). We aim to derive bounds on $\max(N_1^2, N_2^2)$ and $N_1^2 N_2^2$ and used them to find the lower bound on minimum distance in (64).

The mixed input parameters are given in (63). We have

$$\begin{aligned}
\max(N_1^2, N_2^2) - 1 & \leq \max(S_{1,a,t}, S_{1,b,t}) \\
& \leq \max \left(\frac{1+l+\frac{S}{1+l}}{1+\frac{S}{1+l}}, \frac{1+S}{1+l+\frac{S}{1+l}} \right) - 1 \\
& \stackrel{\text{from (55)}}{\leq} \frac{1+l+\frac{S}{1+l}}{1+\frac{S}{1+l}} - 1
\end{aligned}$$

$$\begin{aligned}
&= \frac{l}{1 + \frac{S}{1+l}} \\
&\leq l = \min(S, l),
\end{aligned} \tag{120}$$

and

$$\begin{aligned}
N_1^2 N_2^2 - 1 &\leq (S_{1,a,t} + 1)(S_{1,b,t} + 1) - 1 \\
&\stackrel{\text{from (63)}}{\leq} \left(\frac{1 + l + \frac{S}{1+l}}{1 + \frac{S}{1+l}} \right) \left(\frac{1 + S}{1 + l + \frac{S}{1+l}} \right) - 1 \\
&= \frac{1 + S}{1 + \frac{S}{1+l}} - 1 \\
&= \frac{l \frac{S}{1+l}}{1 + \frac{S}{1+l}} \\
&\leq l.
\end{aligned} \tag{121}$$

Recall the definition of $\kappa_{\gamma, N_1, N_2}$ in (58e). By plugging the bounds in (120)-(121) into (58) we get

$$\begin{aligned}
&\min_{i \in [1:2]} \frac{d_{\min(S_i)}^2}{12 \kappa_{\gamma, N_1, N_2}^2} \\
&\geq \frac{(1 - \max(\delta_1, \delta_2)) \min \left(\frac{l}{\max(N_1^2, N_2^2) - 1}, \frac{S}{N_1^2 N_2^2 - 1} \right)}{1 + (S + l) \max(\delta_1, \delta_2)} \\
&\geq \frac{l}{1 + S + 2l} \min \left(\frac{l(1 + S + l)}{l(1 + l)}, \frac{S(1 + l + S)}{Sl} \right) \\
&= \frac{1 + S + l}{1 + S + 2l} \cdot \frac{l}{1 + l} \\
&\stackrel{1 \leq l \leq S}{\geq} \frac{1 + 2l}{1 + 3l} \cdot \frac{l}{1 + l} \geq \frac{3}{8}.
\end{aligned} \tag{122}$$

Note that the above derivation assumes $1 \leq l$; this restriction is without loss of generality since for $l \leq 1$ TIN with Gaussian codebooks is optimal to within 1/2 bit [23]. Note also that the minimum distance lower bound holds up to an outage set of measure less than γ , where γ is a tunable parameter; the reason why we need an outage set in this regime is the same as in Remark 6.

B. Gap Derivation for $\mathcal{R}_{2R_1+R_2}$ for Regime Weak 1

Outer Bound $\mathcal{R}_{2R_1+R_2}$: With the corner point expressions in (61) we write the outer bound in (54b) as

$$\mathcal{R}_{2R_1+R_2}^{(\text{IV-D})} = \quad (123)$$

$$\bigcup_{t \in [0,1]} \left\{ \begin{array}{l} R_1 \leq \frac{t}{2} \log \left(\frac{1+I+\frac{S}{1+I}}{1+\frac{S}{1+I}} \right) + \frac{1-t}{2} \log \left(\frac{1+S}{1+\frac{S}{1+I}} \right) \\ \quad + \frac{1}{2} \log \left(1 + \frac{S}{1+I} \right) =: I_g(S_{2,a,t}) + I_g\left(\frac{S}{1+I}\right) \\ R_2 \leq \frac{t}{2} \log \left(\frac{1+I+S}{1+I+\frac{S}{1+I}} \cdot \frac{1+S}{1+I} \right) + (1-t)c \\ \quad =: I_g(S_{2,b,t}) + \frac{t}{2} \log \left(\frac{1+S}{1+I} \right) + (1-t)c \end{array} \right\}, \quad (124)$$

where $(1-t)c \leq c \leq \log(2)$, where the parameter c is defined in (53g).

Inner Bound for $\mathcal{R}_{2R_1+R_2}$: In order to approximately achieve the points in (124) we pick

$$N_1 = N_d(S_{2,a,t}), \quad (125a)$$

$$S_{2,a,t} := \left(\frac{1+I+\frac{S}{1+I}}{1+\frac{S}{1+I}} \right)^t \left(\frac{1+S}{1+\frac{S}{1+I}} \right)^{1-t} - 1, \quad (125b)$$

$$N_2 = N_d(S_{2,b,t}), \quad S_{2,b,t} := \left(\frac{1+I+S}{1+I+\frac{S}{1+I}} \right)^t - 1, \quad (125c)$$

$$\delta_1 = \frac{1}{1+I}, \quad (125d)$$

$$\begin{aligned} \delta_2 : I_g(S\delta_2) &= \frac{t}{2} \log \left(\frac{1+S}{1+I} \right) \\ \iff \delta_2 &= \left(\left(\frac{1+S}{1+I} \right)^t - 1 \right) \frac{1}{S}, \end{aligned} \quad (125e)$$

where the power split δ_2 in (125e) satisfies

$$\delta_2 \leq \frac{1-I/S}{1+I} \leq \frac{1}{1+I},$$

as required for the achievable rate region in (57).

Gap for $\mathcal{R}_{2R_1+R_2}$: The gap between the outer bound region in (124) and the achievable rate region in (57) with the choice in (125) is

$$\begin{aligned} \Delta_{R_1} &= I_g(S_{2,a,t}) + I_g\left(\frac{S}{1+I}\right) - \log(N_d(S_{2,a,t})) \\ &\quad - I_g\left(\frac{S}{1+I}\right) + \Delta_{(57)} \end{aligned}$$

$$\leq \log(2) + \Delta_{(57)},$$

and similarly

$$\begin{aligned} \Delta_{R_2} &= I_g(S_{2,b,t}) + \frac{t}{2} \log\left(\frac{1+S}{1+l}\right) + (1-t)c \\ &\quad - \log(N_d(S_{2,b,t})) - \frac{t}{2} \log\left(\frac{1+S}{1+l}\right) + \Delta_{(57)} \\ &\leq \log(2) + \log(2) + \Delta_{(57)}, \end{aligned}$$

since $(1-t)c \leq c \leq \log(2)$, where the parameter c is defined in (53g).

So we are left with bounding $\Delta_{(57)}$ in (57), which is related to the minimum distance of the received constellations S_1 and S_2 defined in (35). In Appendix E-C we show that

$$\min_{i \in [1:2]} \frac{d_{\min}^2(S_i)}{12} \geq \kappa_{\gamma, N_1, N_2}^2 \cdot \frac{4}{45}, \quad (126)$$

where $\kappa_{\gamma, N_1, N_2}$ is given in (58e), and $\max(N_1^2, N_2^2) - 1 \leq l = \min(S, l)$. With this, we finally get that the gap for this face is bounded by

$$\begin{aligned} \text{gap}_{(127)} &\leq \max(\Delta_{R_1}, \Delta_{R_2}) \\ &\leq \frac{1}{2} \log\left(\frac{16\pi e}{3}\right) + \frac{1}{2} \log\left(1 + \frac{45}{4} \cdot \frac{1}{\kappa_{\gamma, N_1, N_2}^2}\right) \\ &\leq \text{Gap}(16, 45). \end{aligned} \quad (127)$$

C. Derivation of (126)

We aim to derive different bounds involving N_1^2 and N_2^2 and used them in the minimum distance lower bound in (58).

From (125a) we have

$$\begin{aligned} N_1^2 - 1 &\leq S_{2,a,t} \leq \max(S_{2,a,0}, S_{2,a,1}) \\ &\leq \frac{\max\left(1+l+\frac{S}{1+l}, 1+S\right)}{1+\frac{S}{1+l}} - 1 \\ &\stackrel{\text{for } 1+l \leq S}{\leq} \frac{1+S}{1+\frac{S}{1+l}} - 1 \\ &= l \cdot \frac{\frac{S}{1+l}}{1+\frac{S}{1+l}} \leq l, \end{aligned} \quad (128)$$

from (125c) we have

$$\begin{aligned}
N_2^2 - 1 &\leq S_{2,b,t} \leq \max(S_{2,b,0}, S_{2,b,1}) \\
&\leq \frac{1 + l + S}{1 + l + \frac{S}{1+l}} - 1 \\
&= \frac{l \cdot \frac{S}{1+l}}{1 + l + \frac{S}{1+l}} \\
&\leq \min\left(l, \frac{S}{1+l}\right) = \frac{S}{1+l},
\end{aligned} \tag{129}$$

finally

$$\max(N_1^2, N_2^2) - 1 \leq \max\left(l, \frac{S}{1+l}\right) = l = \min(S, l). \tag{130}$$

We also have

$$\begin{aligned}
N_1^2 N_2^2 - 1 &\leq (S_{2,a,t} + 1)(S_{2,b,t} + 1) - 1 \\
&= \frac{(1 + S)^{1-t} (1 + l + S)^t}{1 + \frac{S}{1+l}} - 1 \\
&\leq \frac{1 + l + S}{1 + \frac{S}{1+l}} - 1 = l.
\end{aligned} \tag{131}$$

In this regime, as we shall soon see, it also important to bound

$$\begin{aligned}
(1 + S\delta_2)(1 + S_{2,a,t}) &= \frac{1 + S}{1 + \frac{S}{1+l}} \left(\frac{1 + l + \frac{S}{1+l}}{1 + l} \right)^t \\
&\leq \frac{1 + S}{1 + S + l} \left(1 + l + \frac{S}{1+l} \right).
\end{aligned} \tag{132}$$

We next bound the minimum distances. Recall that $\kappa_{\gamma, N_1, N_2}$ is given in (58e).

With (58c) we have

$$\begin{aligned}
&\frac{d_{\min(S_1)}^2}{12 \kappa_{\gamma, N_1, N_2}^2} \\
&\geq \frac{1 - \max(\delta_1, \delta_2)}{1 + S\delta_1 + l\delta_2} \min\left(\frac{l}{N_2^2 - 1}, \frac{S}{N_1^2 N_2^2 - 1}\right) \\
&\stackrel{(a)}{\geq} \frac{l}{1 + S + 2l} \min\left(\frac{l}{N_2^2 - 1}, \frac{S}{N_1^2 N_2^2 - 1}\right) \\
&\stackrel{(b)}{\geq} \frac{l}{1 + S + 2l} \min\left(\frac{l(1+l)}{S}, \frac{S}{l}\right) \\
&= \min\left(\frac{l^2(1+l)}{S(1+S+2l)}, \frac{S}{1+S+2l}\right)
\end{aligned}$$

$$\begin{aligned}
&\stackrel{(c)}{\geq} \min \left(\frac{l^2 (1+l) \left(1 + \frac{S}{1+l}\right)}{(1+S+2l) \left(1+l + \frac{S}{1+l}\right)^2}, \frac{S}{1+S+2l} \right) \\
&\stackrel{(d)}{\geq} \min \left(\frac{l^2 (1+l+S)}{(1+S+2l)(1+2l)^2}, \frac{1+l}{2+3l} \right) \\
&\stackrel{(e)}{\geq} \min \left(\frac{2l^2 (1+l)}{(2+3l)(1+2l)^2}, \frac{1+l}{2+3l} \right) \\
&\stackrel{(f)}{\geq} \min \left(\frac{4}{45}, \frac{1}{3} \right) = \frac{4}{45},
\end{aligned} \tag{133}$$

where the inequalities follow since: (a) $\delta_2 \leq \delta_1 = \frac{1}{1+l}$, (b) from (129) and (131), (c) from (56) we have $S \leq 1+S \leq \frac{(1+l+\frac{S}{1+l})^2}{1+\frac{S}{1+l}}$, and (d) where we have used $1+l \leq S$, $\frac{S}{1+l} \leq l$ for 1st term must use largest $\frac{S}{1+l}$ while for 2nd smallest S which $1+l$, (e) since $1+l \leq S$, (f) comes from using $1 \leq l$.

With (58d), and recalling that $1+S\delta_2 = \left(\frac{1+S}{1+l}\right)^t$ from (125e) and $S_{2,a,t}$ in (125a), we have

$$\begin{aligned}
&\frac{d_{\min(S_2)}^2}{12 \kappa_{\gamma, N_1, N_2}^2} \\
&\geq \frac{1 - \max(\delta_1, \delta_2)}{1 + S\delta_2 + l\delta_1} \min \left(\frac{l}{N_1^2 - 1}, \frac{S}{N_1^2 N_2^2 - 1} \right) \\
&\stackrel{(a)}{\geq} \frac{\frac{1}{1+l}}{1 + S\delta_2 + \frac{1}{1+l}} \min \left(\frac{l}{S_{2,a,t}}, \frac{S}{l} \right) \\
&\stackrel{(b)}{\geq} \min \left(\frac{\frac{l^2}{1+l}}{2(1+S\delta_2)(1+S_{2,a,t})}, \frac{\frac{S}{1+l}}{1 + \frac{S}{1+l} + \frac{1}{1+l}} \right) \\
&\stackrel{(c)}{\geq} \min \left(\frac{\frac{l^2}{1+l}(1+S+l)}{(1+S)(1+l + \frac{S}{1+l})}, \frac{S}{1+S+2l} \right) \\
&\stackrel{(d)}{\geq} \min \left(\frac{l^2(1+S+l)}{(1+l)(1+S)(1+2l)}, \frac{S}{1+S+2l} \right) \\
&\stackrel{(e)}{\geq} \min \left(\frac{l^2(1+l)^2}{(1+l)(1+l+l^2)(1+2l)}, \frac{1+l}{2+3l} \right) \\
&\stackrel{(f)}{\geq} \min \left(\frac{2}{9}, \frac{1}{3} \right) = \frac{2}{9},
\end{aligned} \tag{134}$$

where the inequalities follow since: (a) $\delta_2 \leq \delta_1 = \frac{1}{1+l}$, $N_1^2 - 1 \leq S_{2,a,t}$ and (131); (b) $(1+S\delta_2) + \frac{1}{1+l} \leq 2+S\delta_2 \leq 2(1+S\delta_2)$ and $\delta_2 \leq \frac{1}{1+l}$, and the rest of the inequalities from the definition of weak interference $1 \leq l$, $1+l \leq S \leq l(1+l)$; (c) from (132); (d) since $\frac{S}{1+l} \leq l$; (e) where we have used $1+l \leq S$, $\frac{S}{1+l} \leq l$ for 1st term must use largest $\frac{S}{1+l}$ while for 2nd smallest S which $1+l$; and (f) comes from using $1 \leq l$.

By putting together (133) and (134), we obtain (126).

APPENDIX F

AUXILIARY RESULTS FOR REGIME WEAK2

We derive here some auxiliary results for the regime in (56), namely

$$(1 + l) \leq S \leq l(1 + l),$$

$$\frac{1 + S}{1 + l + \frac{S}{1+l}} \geq \frac{1 + l + \frac{S}{1+l}}{1 + \frac{S}{1+l}}.$$

A. Derivation of (71)

We aim to derive different bounds on N_1^2, N_2^2, δ_1 and δ_1 so as to obtaine in the minimum distance lower bound in (71).

The mixed input parameters are in (70). We have

$$\begin{aligned} & N_1^2 - 1 \\ & \leq S_{3,a,t} \\ & \leq \max(S_{3,a,0}, S_{3,a,1}) \\ & = \max\left(\frac{(1 + \frac{S}{1+l})(1 + S)}{1 + l + \frac{S}{1+l}}, \frac{(1 + l + \frac{S}{1+l})^3}{(1 + \frac{S}{1+l})(1 + S)}\right)^{\frac{1}{2}} - 1 \\ & \stackrel{\text{from (56)}}{=} \left(1 + \frac{S}{1+l}\right) \sqrt{\frac{1 + S}{(1 + l + \frac{S}{1+l})(1 + \frac{S}{1+l})}} - 1 \\ & \leq \frac{S}{1+l}. \end{aligned} \tag{135}$$

Similarly we have

$$N_2^2 - 1 \leq S_{3,b,t} \leq \frac{S}{1+l}. \tag{136}$$

The bounds in (135)-(136) imply

$$\max(\delta_1, \delta_2) \leq \frac{\max(S_{3,a,t}, S_{3,b,t})}{S} \leq \frac{1}{1+l}. \tag{137}$$

Finally we have

$$\max(N_1^2, N_2^2) - 1 \leq \frac{S}{1+l} \leq l = \min(S, l), \tag{138}$$

by the definition of this regime.

We also have

$$\begin{aligned}
N_1^2 N_2^2 - 1 &\leq (1 + S_{3,a,t})(1 + S_{3,b,t}) - 1 \\
&= (1 + S\delta_1)(1 + S_{3,b,t}) - 1 \\
&= (1 + S_{3,a,t})(1 + S\delta_2) - 1 \\
&= l + \frac{S}{1+l} \leq 2l.
\end{aligned} \tag{139}$$

With (58c) we have

$$\begin{aligned}
&\frac{d_{\min(S_1)}^2}{12 \kappa_{\gamma, N_1, N_2}^2} \\
&\geq \frac{1 - \max(\delta_1, \delta_2)}{1 + S\delta_1 + l\delta_2} \min\left(\frac{l}{N_2^2 - 1}, \frac{S}{N_1^2 N_2^2 - 1}\right) \\
&\stackrel{(a)}{\geq} \frac{\frac{l}{1+l}}{1 + S_{3,a,t} + \frac{l}{1+l}} \min\left(\frac{l}{S_{3,b,t}}, \frac{S}{2l}\right) \\
&\stackrel{(b)}{\geq} \min\left(\frac{\frac{l^2}{1+l}}{2(1+l + \frac{S}{1+l})}, \frac{S\frac{l}{1+l}}{(1 + \frac{S}{1+l} + \frac{l}{1+l})2l}\right) \\
&= \min\left(\frac{l^2}{2[(1+l)^2 + S]}, \frac{S}{2[1 + 2l + S]}\right) \\
&\stackrel{(c)}{\geq} \min\left(\frac{l^2}{2[(1+l)^2 + l(1+l)]}, \frac{l(1+l)}{2[1 + 2l + l(1+l)]}\right) \\
&\stackrel{(c)}{\geq} \min\left(\frac{l^2}{2[(1+l)^2 + l(1+l)]}, \frac{1+l}{2[2 + 3l]}\right) \\
&= \min\left(\frac{1}{12}, \frac{1}{6}\right) = \frac{1}{12},
\end{aligned} \tag{140}$$

where the inequalities follow from: (a) using (70e), (137) and (139); (b) using (139); and (c) since $1 \leq l$ and $1 + l \leq S \leq l(1 + l)$.

By symmetry an equivalent bound can be derived for $d_{\min(S_2)}^2$.

Hence minimum distance in (71) is bounded by

$$\min_{i \in [1:2]} \frac{d_{\min(S_i)}^2}{12} \geq \kappa_{\gamma, N_1, N_2}^2 \frac{1}{12}. \tag{141}$$

B. Gap for $\mathcal{R}_{2R_1+R_2}$ using common \rightarrow discrete map

Outer Bound $\mathcal{R}_{2R_1+R_2}$: With the corner point expressions in (68) we write the outer bound in (54b) as

$$\mathcal{R}_{2R_1+R_2}^{(\text{IV-E})} =$$

$$\bigcup_{t \in [0,1]} \left\{ \begin{array}{l} R_1 \leq \frac{1-t}{2} \log \left(\frac{(1+\frac{S}{1+I})(1+S)}{1+I+\frac{S}{1+I}} \right) + \frac{t}{2} \log(1+S) \\ \quad =: I_g(S_{4,a,t}) + I_g\left(\frac{S}{1+I}\right) \\ R_2 \leq \frac{1-t}{2} \log \left(\frac{(1+I+\frac{S}{1+I})^3(1+I)}{1+S} \right) + tc \\ \quad =: I_g(S_{4,b,t}) + \frac{1-t}{2} \log \left(\frac{(1+I+\frac{S}{1+I})(1+I)}{1+I+S} \right) \\ \quad + tc \end{array} \right\}, \quad (142)$$

where $tc \leq c \leq \log(2)$, where the parameter c is defined in (53g), and $0 \leq t \leq 1$.

Inner Bound for $\mathcal{R}_{2R_1+R_2}$: In order to approximately achieve the points in $\mathcal{R}_{2R_1+R_2}$ in (142) we pick

$$N_1 = N_d(S_{4,a,t}), \quad (143a)$$

$$S_{4,a,t} := \frac{1+S}{\left(1+\frac{S}{1+I}\right)^t \left(1+I+\frac{S}{1+I}\right)^{1-t}} - 1, \quad (143b)$$

$$N_2 = N_d(S_{4,b,t}), \quad S_{4,b,t} := \left(\frac{(1+I+\frac{S}{1+I})^2}{1+S} \right)^{1-t} - 1, \quad (143c)$$

$$\delta_1 := \frac{1}{1+I}, \quad (143d)$$

$$\begin{aligned} \delta_2 : I_g(S\delta_2) &= \frac{1-t}{2} \log \left(\frac{1+I+\frac{S}{1+I}}{1+\frac{S}{1+I}} \right) \\ &\iff \delta_2 = \left(\left(\frac{1+I+\frac{S}{1+I}}{1+\frac{S}{1+I}} \right)^{1-t} - 1 \right) \frac{1}{S}, \end{aligned} \quad (143e)$$

where in Appendix F-C, eq.(149), we show that

$$\delta_2 \leq \delta_1 = \frac{1}{1+I},$$

as required for the achievable region in (57).

Gap for $\mathcal{R}_{2R_1+R_2}$: The gap between the outer bound in (54b) and achievable region in (57) with the parameters in (143) is

$$\begin{aligned} \Delta_{R_1} &= I_g(S_{4,a,t}) + I_g\left(\frac{S}{1+I}\right) - \log(N_d(S_{4,a,t})) \\ &\quad - I_g(S\delta_1) + \Delta_{(57)} \\ &\leq \log(2) + \Delta_{(57)}, \end{aligned}$$

and similarly we have that

$$\begin{aligned}\Delta_{R_2} &= \mathsf{I}_g(\mathsf{S}_{4,b,t}) + \frac{1-t}{2} \log \left(\frac{(1+\mathsf{I} + \frac{\mathsf{S}}{1+\mathsf{I}})(1+\mathsf{I})}{1+\mathsf{I}+\mathsf{S}} \right) + tc \\ &\quad - \log(\mathsf{N}_d(\mathsf{S}_{4,b,t})) - \mathsf{I}_g(\mathsf{S}\delta_2) + \Delta_{(57)} \\ &\leq \log(2) + \log(2) + \Delta_{(57)},\end{aligned}$$

since $tc \leq c \leq \log(2)$, where the parameter c is defined in (53g), and $0 \leq t \leq 1$.

So, we are left with bounding $\Delta_{(57)}$ which is related to the minimum distances of the sum-set constellations. In Appendix F-C we show that

$$\min_{i \in [1:2]} \frac{d_{\min(S_i)}^2}{12} \geq \kappa_{\gamma, N_1, N_2}^2 \cdot \frac{1}{8}, \quad (144)$$

where $\kappa_{\gamma, N_1, N_2}$ is given in (58e), and that $\max(N_1^2, N_2^2) - 1 \leq \mathsf{I}$; with this, we finally get that the gap for this face is bounded by

$$\begin{aligned}\text{gap}_{(145)} &\leq \max(\Delta_{R_1}, \Delta_{R_2}) = 2\log(2) + \Delta_{(57)} \\ &\leq \text{Gap}(16, 32).\end{aligned} \quad (145)$$

C. Proof of (144)

We first derive some bounds on N_1^2 and N_2^2 that will be useful in bounding minimum distance of the received constellations.

From (143a) we have

$$\begin{aligned}N_1^2 - 1 &\leq \mathsf{S}_{4,a,t} \leq \max(\mathsf{S}_{4,a,0}, \mathsf{S}_{4,a,1}) \\ &= \frac{1+\mathsf{S}}{\min\left(1 + \frac{\mathsf{S}}{1+\mathsf{I}}, 1 + \mathsf{I} + \frac{\mathsf{S}}{1+\mathsf{I}}\right)} - 1 \\ &= \frac{1+\mathsf{S}}{1 + \frac{\mathsf{S}}{1+\mathsf{I}}} - 1 \\ &= \mathsf{I} \cdot \frac{\frac{\mathsf{S}}{1+\mathsf{I}}}{1 + \frac{\mathsf{S}}{1+\mathsf{I}}} \leq \mathsf{I}.\end{aligned} \quad (146)$$

Similarly from (143c)

$$\begin{aligned}N_2^2 - 1 &\leq \mathsf{S}_{4,b,t} \leq \max(\mathsf{S}_{4,b,0}, \mathsf{S}_{4,b,1}) \\ &= \frac{(1+\mathsf{I} + \frac{\mathsf{S}}{1+\mathsf{I}})^2}{1+\mathsf{S}} - 1\end{aligned}$$

$$\leq \frac{(1+S)(1+\frac{S}{1+l})}{1+S} - 1 = \frac{S}{1+l}, \quad (147)$$

where inequality follow the definition of the regime in (56). We also have

$$\begin{aligned} N_1^2 N_2^2 - 1 &\leq (1 + S_{4,a,t})(1 + S_{4,b,t}) - 1 \\ &= \left(1 + l + \frac{S}{1+l}\right)^{1-t} \left(\frac{1+S}{1+\frac{S}{1+l}}\right)^t - 1 \\ &\leq \max\left(l + \frac{S}{1+l}, l\frac{\frac{S}{1+l}}{1+\frac{S}{1+l}}\right) \\ &= l + \frac{S}{1+l} \leq 2l, \end{aligned} \quad (148)$$

where the last inequality follows from $\frac{S}{1+l} \leq l$.

From (143e) we have

$$\begin{aligned} S\delta_2 &\leq \frac{1+l+\frac{S}{1+l}}{1+\frac{S}{1+l}} - 1 \\ &\stackrel{(a)}{\leq} \frac{1+S}{1+l+\frac{S}{1+l}} - 1 \\ &= \frac{l\frac{S}{1+l} - l}{1+l+\frac{S}{1+l}} \\ &\leq \frac{l\frac{S}{1+l}}{1+l+\frac{S}{1+l}} \\ &\leq \min\left(l, \frac{S}{1+l}\right) \\ &\stackrel{(b)}{=} \frac{S}{1+l}, \end{aligned} \quad (149)$$

where inequalities follow from: (a) using definition of the regime in (56); and (b) using $\frac{S}{1+l} \leq l$.

As for the derivation in Section E-C, another key bound is

$$\begin{aligned} &(1 + S\delta_1)(1 + S_{4,b,t}) \\ &\leq \left(1 + \frac{S}{1+l}\right) \left(\frac{(1+l+\frac{S}{1+l})^2}{1+S}\right)^{1-t} \\ &\leq \left(1 + \frac{S}{1+l}\right) \frac{(1+l+\frac{S}{1+l})^2}{1+S} \\ &\stackrel{(a)}{\leq} \frac{1+l+S}{1+S} \cdot \frac{(1+2l)^2}{1+l}, \end{aligned} \quad (150)$$

where inequalities follow from: (a) using $\frac{S}{1+l} \leq l$; and (b) using $\frac{1+l+S}{1+S} \leq 2$ and $\frac{1+2l}{1+l} \leq 2$.

Similarly, we have

$$\begin{aligned}
& (1 + S_{4,a,t})(1 + S\delta_2) \\
& \leq \frac{1 + S}{\left(1 + \frac{S}{1+l}\right)^t \left(1 + l + \frac{S}{1+l}\right)^{1-t}} \left(\frac{1 + l + \frac{S}{1+l}}{1 + \frac{S}{1+l}}\right)^{1-t} \\
& = \frac{1 + S}{1 + \frac{S}{1+l}} \leq 1 + l.
\end{aligned} \tag{151}$$

By using (58c), the minimum distance for S_1 can be bounded as

$$\begin{aligned}
& \frac{d_{\min(S_1)}^2}{12 \kappa_{\gamma, N_1, N_2}^2} \\
& \geq \frac{1 - \max(\delta_1, \delta_2)}{1 + S\delta_1 + l\delta_2} \min\left(\frac{l}{N_2^2 - 1}, \frac{S}{N_1^2 N_2^2 - 1}\right) \\
& \stackrel{(a)}{\geq} \frac{\frac{l}{1+l}}{1 + S\delta_1 + l\delta_2} \min\left(\frac{l}{S_{4,b,t}}, \frac{S}{2l}\right) \\
& \stackrel{(b)}{\geq} \min\left(\frac{\frac{l^2}{1+l}}{2(1 + S\delta_1)(1 + S_{4,b,t})}, \frac{S}{2(1 + S + 2l)}\right) \\
& \stackrel{(c)}{\geq} \min\left(\frac{l^2(1 + S)}{2(1 + l + S)(1 + 2l)^2}, \frac{1 + l}{2(2 + 3l)}\right) \\
& \stackrel{(d)}{\geq} \min\left(\frac{3}{8}, \frac{1}{6}\right) = \frac{1}{6},
\end{aligned}$$

where inequalities follow from: (a) $\max(\delta_1, \delta_2) \leq \frac{1}{1+l}$ and from (147) and (148) we have that $N_2^2 - 1 \leq S_{4,b,t}$ and $N_1^2 N_2^2 - 1 \leq 2l$; (b) from (149) $\delta_2 \leq \frac{1}{1+l}$; (c) using (149) we have $S_{4,b,t}(2 + S\delta_1) \leq 2(1 + S_{4,b,t})(1 + S\delta_1)$ and (150); and (d) $S \geq (1 + l)$ and $l \geq 1$.

Similarly,

$$\begin{aligned}
& \frac{d_{\min(S_2)}^2}{12 \kappa_{\gamma, N_1, N_2}^2} \\
& \geq \frac{1 - \max(\delta_1, \delta_2)}{1 + S\delta_2 + l\delta_1} \min\left(\frac{l}{N_1^2 - 1}, \frac{S}{N_1^2 N_2^2 - 1}\right) \\
& \stackrel{(a)}{\geq} \min\left(\frac{\frac{l^2}{1+l}}{2(1 + S\delta_2)(1 + S_{4,a,t})}, \frac{S}{2(1 + S + 2l)}\right) \\
& \stackrel{(b)}{\geq} \min\left(\frac{l^2}{2(1 + l)(1 + l)}, \frac{1 + l}{2(2 + 3l)}\right) \\
& \stackrel{(c)}{\geq} \min\left(\frac{1}{8}, \frac{1}{6}\right) = \frac{1}{8},
\end{aligned}$$

where inequalities follow from: (a) (143a) we have $N_1^2 - 1 \leq S_{4,a,t}$, from (149) and (143d) $\max(\delta_1, \delta_2) \leq \frac{1}{1+l}$ and from (148) $N_1^2 N_2^2 - 1 \leq l$; (b) $S_{4,a,t}(2 + S\delta_2) \leq 2(1 + S_{4,a,t})(1 + S\delta_2)$ and (151); and (c) from $S \geq (1 + l)$ and $l \geq 1$.

Hence, the minimum distance in (144) is bounded by

$$\min_{i \in [1:2]} \frac{d_{\min(S_i)}^2}{12 \kappa_{\gamma, N_1, N_2}^2} \geq \frac{1}{8}.$$

APPENDIX G

CONSTANT GAP DERIVATION FOR REGIME WEAK2

A. Another Inner Bound for $\mathcal{R}_{R_1+R_2}$

In order to approximately achieve the points in $\mathcal{R}_{R_1+R_2}^{(\text{IV-E})}$ in (69) we pick

$$N_1 = N_d \left(\frac{1}{k} S_{3,a,t} \right), \quad S_{3,a,t} \text{ in (70a)} \quad (152a)$$

$$N_2 = N_d \left(\frac{1}{k} S_{3,b,t} \right), \quad S_{3,b,t} \text{ in (70c)} \quad (152b)$$

$$\delta_1 : l_g(S\delta_1) = l_g(S_{3,a,t}) \iff \delta_1 = \frac{S_{3,a,t}}{S}, \quad (152c)$$

$$\delta_2 : l_g(S\delta_2) = l_g(S_{3,b,t}) \iff \delta_2 = \frac{S_{3,b,t}}{S}. \quad (152d)$$

where k is a parameter that we will tune in order to satisfy the non-overlap condition in Proposition 2. Indeed, in order to check whether we can use the bound in (59a) we must check whether the condition in (59b) holds. To simplify the analytical computations we choose to satisfy instead

$$\frac{(1 - \delta_{i'}) N_{i'}^2}{N_{i'}^2 - 1} \leq k \leq \frac{S}{l} \frac{(1 - \delta_i)}{N_i^2 - 1} \quad \forall (i, i') \in \{(1, 2), (2, 1)\},$$

for some k ; since $\frac{(1 - \delta_{i'}) N_{i'}^2}{N_{i'}^2 - 1} \leq \frac{N_{i'}^2}{N_{i'}^2 - 1} \leq \frac{4}{3}$ for all $N_{i'} \geq 2$, we set $\frac{4}{3} := k$. In other words, we accept an increase in gap of $\log(k) = \log(4/3)$, due to the reduction of the number of points of the discrete part of the mixed inputs from $N_d(x)$ to $N_d(3x/4)$ for some ‘SNR’ x , for ease of computations. Therefore, for the rest of this section instead of checking condition in (59b) we will check the simpler condition

$$\frac{4}{3} l \leq \frac{S(1 - \delta_i)}{N_i^2 - 1} \quad \forall i \in [1 : 2]. \quad (153)$$

The gap between the outer bound region in (69) and the achievable rate in (57) with the parameters in (152) is

$$\Delta_{R_1} = 2l_g(S_{3,a,t}) - \log(N_d(3/4 S_{3,a,t})) - l_g(S_{3,a,t})$$

$$\begin{aligned}
& + \Delta_{(57)} \\
& \leq \log \left(\frac{8}{3} \right) + \Delta_{(57)},
\end{aligned}$$

and similarly

$$\Delta_{R_2} \leq \log \left(\frac{8}{3} \right) + \Delta_{(57)}.$$

We are then left with bounding $\Delta_{(57)}$, which depends on minimum distances of the received sum-set constellations. From (135)-(136) we have

$$\begin{aligned}
N_1^2 - 1 & \leq \frac{3}{4} S_{3,a,t} \leq \frac{3}{4} \frac{S}{1+l}, \quad \text{from (135),} \\
N_2^2 - 1 & \leq \frac{3}{4} S_{3,b,t} \leq \frac{3}{4} \frac{S}{1+l}, \quad \text{from (136),}
\end{aligned}$$

and thus

$$\frac{S(1 - \delta_i)}{N_i^2 - 1} \stackrel{\max(\delta_1, \delta_2) \leq \frac{1}{1+l}}{\geq} l \frac{\frac{S}{1+l}}{N_i^2 - 1} \geq \frac{4}{3} l, \quad (154)$$

as needed in (153).

Therefore, by (59a), for $d_{\min(S_1)}^2$ we have that

$$\begin{aligned}
\frac{d_{\min(S_1)}^2}{12} & = \frac{1}{1 + S\delta_1 + l\delta_2} \min \left(\frac{(1 - \delta_1)S}{N_1^2 - 1}, \frac{(1 - \delta_2)l}{N_2^2 - 1} \right) \\
& \stackrel{(a)}{\geq} \frac{\frac{1}{1+l}}{1 + S\delta_1 + \frac{1}{1+l}} \min \left(\frac{S}{N_1^2 - 1}, \frac{l}{N_2^2 - 1} \right) \\
& \stackrel{(b)}{\geq} \frac{\frac{1}{1+l}}{1 + S\delta_1 + \frac{1}{1+l}} \frac{4}{3} \min \left(1 + l, \frac{l}{S_{3,b,t}} \right) \\
& \stackrel{(c)}{\geq} \frac{4}{3} \min \left(\frac{l(1+l)}{1 + S + 2l}, \frac{\frac{l^2}{1+l}}{2(1 + S\delta_1)(1 + S_{3,b,t})} \right) \\
& \stackrel{(d)}{\geq} \frac{4}{3} \min \left(\frac{l(1+l)}{1 + S + 2l}, \frac{l^2}{2(1+l)(1+l+\frac{S}{1+l})} \right) \\
& \stackrel{(e)}{\geq} \frac{4}{3} \min \left(\frac{l(1+l)}{1 + 3l + l^2}, \frac{l^2}{2(1+l)(1+2l)} \right) \\
& \geq \frac{4}{3} \min \left(\frac{2}{5}, \frac{1}{12} \right) = \frac{1}{9}, \quad (155)
\end{aligned}$$

where the inequalities follows from: (a) $\max(\delta_1, \delta_2) \leq \frac{1}{1+l}$; (b) from (135) and (136); (c) $\max(\delta_1, \delta_2) \leq \frac{1}{1+l}$; (d) from (139); and (e) from $1 \leq l \leq S \leq l(1+l)$.

By symmetry, $\frac{d_{\min(S_2)}^2}{12}$ is bounded in the same way, thus

$$\min_{i \in [1:2]} \frac{d_{\min(S_i)}^2}{12} \geq \frac{1}{9}. \quad (156)$$

Finally the gap for this face is

$$\begin{aligned} \text{gap}_{(157)} &\leq \max(\Delta_{R_1}, \Delta_{R_2}) = \log\left(\frac{8}{3}\right) + \Delta_{(57)} \\ &\leq \log\left(\frac{8}{3}\right) + \frac{1}{2} \log\left(\frac{\pi e}{3}\right) + \frac{1}{2} \log(1+9) \\ &= \frac{1}{2} \log\left(\frac{640\pi e}{27}\right) = \text{Gap}\left(\frac{640}{9}, 0\right) \approx 3.83 \text{ bits}. \end{aligned} \quad (157)$$

B. Another Inner Bound for $\mathcal{R}_{2R_1+R_2}$

We choose the mixed input parameters as

$$N_1 = N_d \left(\frac{3}{4} \frac{S-1}{1+1} \right), \quad (158a)$$

$$N_2 = N_d \left(\frac{3}{4} S_{4,b,t} \right), \quad (158b)$$

$$S_{4,b,t} := \left(\frac{(1+1+\frac{S}{1+1})^2}{1+S} \right)^{1-t} - 1 \stackrel{\text{by eq.(147)}}{\leq} \frac{S}{1+1}, \quad (158c)$$

$$\delta_1 = \frac{S_{4,a,t}}{S} \stackrel{\text{by eq.(146)}}{\leq} \frac{1}{S}, \quad (158d)$$

$$\delta_2 = \frac{1+1+\frac{S}{1+1}}{(1+\frac{S}{1+1})(1+S)} \stackrel{\text{by eq.(56)}}{\leq} \frac{1}{1+1+\frac{S}{1+1}} \leq \frac{1}{1+1}, \quad (158e)$$

where the factor $\frac{3}{4}$ in the number of points appears for the same reason as in Section G-A.

An inequality we will need is

$$\begin{aligned} &\frac{S\delta_2}{1+1\delta_1} \\ &\stackrel{(a)}{\geq} \frac{S^{\frac{(1+1+\frac{S}{1+1})(1+1)}{(1+1+S)(1+S)}}}{\frac{1}{S} \frac{(1+S)}{(1+\frac{S}{1+1})^t (1+1+\frac{S}{1+1})^{1-t}}} \\ &= \frac{S^2}{(1+S)^2} \frac{(1+1+\frac{S}{1+1}) (1+\frac{S}{1+1})^t (1+1+\frac{S}{1+1})^{1-t}}{1(1+\frac{S}{1+1})} \\ &= \frac{S^2}{(1+S)^2} \frac{(1+1+\frac{S}{1+1}) (1+1)^{1-t} (1+1+\frac{S}{1+1})^{1-t}}{1(1+1+S)^{1-t}} \\ &\stackrel{(b)}{\geq} \frac{3}{4} \left(\frac{1+1+\frac{S}{1+1}}{1+\frac{S}{1+1}} \right)^{1-t}, \end{aligned} \quad (159)$$

where the inequalities follow from: (a) plugin in values of δ_1 and δ_2 and lower bounding the denominator; and (b) using $S \geq 1$ we have that $\frac{S^2}{(1+S)^2} \geq \frac{1}{4}$ and using $S \geq (1+l)$ we have $\frac{1+l+\frac{S}{1+l}}{1} \geq \frac{2+l}{1} \geq 3$.

Another inequality we will need is

$$\begin{aligned}
 S\delta_2 &= S \frac{(1+l+\frac{S}{1+l})(1+l)}{(1+l+S)(1+S)} \\
 &\leq \frac{(1+l+\frac{S}{1+l})(1+l)}{(1+l+S)} \\
 &= \frac{(1+l+\frac{S}{1+l})}{(1+\frac{S}{1+l})} \\
 &\leq \frac{(1+2l)(1+l)}{S},
 \end{aligned} \tag{160}$$

where the last inequality comes from using $\frac{S}{1+l} \leq l$ and dropping one in the denominator.

Gap for $\mathcal{R}_{2R_1+R_2}$: The gap between the outer bound in (142) and the achievable rate in Proposition 5 with the choice of parameters in (158) is

$$\begin{aligned}
 \Delta_{R_1} &= I_g(S_{4,a,t}) + I_g\left(\frac{S}{1+l}\right) - \log\left(N_d\left(\frac{3}{4} \frac{S-l}{1+l}\right)\right) \\
 &\quad - I_g(S\delta_1) + \Delta_{(57)} \\
 &\leq \log(2) + \frac{1}{2} \log(2) + \Delta_{(57)},
 \end{aligned}$$

where

$$\begin{aligned}
 &I_g\left(\frac{S}{1+l}\right) - I_g\left(\frac{3}{4} \frac{S-l}{1+l}\right) \\
 &= \frac{1}{2} \log \frac{1+l+S}{1+l} \frac{1+l}{1+l/4+3S/4} \\
 &\leq \frac{1}{2} \log \frac{1+2S}{1+S} \leq \frac{1}{2} \log(2),
 \end{aligned}$$

and similarly

$$\begin{aligned}
 \Delta_{R_2} &= I_g(S_{4,b,t}) + \frac{1-t}{2} \log\left(\frac{(1+l+\frac{S}{1+l})(1+l)}{1+l+S}\right) \\
 &\quad + tc - \log(N_d(S_{4,b,t})) - I_g \log\left(\frac{S\delta_2}{1+l\delta_1}\right) \\
 &\quad - \frac{1}{2} \log(2) + \Delta_{(57)} \\
 &\leq \log(2) + \frac{1}{2} \log\left(\frac{4}{3}\right) + \frac{1}{2} \log\left(\frac{4}{3}\right) + \log(2)
 \end{aligned}$$

$$-\frac{1}{2}\log(2) + \Delta_{(57)} = \frac{1}{2}\log\left(\frac{2^7}{3^2}\right) + \Delta_{(57)},$$

where we have used $tc \leq \log(2)$ and the bound in (159); the term $-\frac{1}{2}\log(2)$ is because of the definition of $\Delta_{(57)}$ that assumed $\max(\delta_1, \delta_2) \leq \frac{1}{1+l}$, which is not the case here.

So, we are left with bounding $\Delta_{(57)}$, which depends on the minimum distances of the received constellations. We must verify the condition in (153) at each receiver.

For receiver 1 we have

$$\begin{aligned} \frac{S(1-\delta_1)}{N_1^2-1} &\stackrel{\text{from eq.(158d)}}{\geq} \frac{S-l}{N_1^2-1} \\ &\stackrel{\text{from eq.(158a)}}{\geq} \frac{S-l}{\frac{3}{4}\frac{S-l}{1+l}} \\ &= \frac{4}{3}(1+l) \geq \frac{4}{3}l, \end{aligned} \tag{161}$$

and therefore

$$\begin{aligned} &\frac{d_{\min(S_1)}^2}{12} \\ &= \frac{1}{1+S\delta_1+l\delta_2} \min\left(\frac{(1-\delta_1)S}{N_1^2-1}, \frac{(1-\delta_2)l}{N_2^2-1}\right) \\ &\stackrel{(a)}{\geq} \frac{1}{1+S\delta_1+l\frac{1}{1+l}} \min\left(\frac{4}{3}(1+l), \frac{\frac{l^2}{1+l}}{\frac{3}{4}S_{4,b,t}}\right) \\ &= \frac{4}{3} \min\left(\frac{(1+l)}{1+S\delta_1+l\frac{1}{1+l}}, \frac{\frac{l^2}{1+l}}{(1+S\delta_1+l\frac{1}{1+l})S_{4,b,t}}\right) \\ &\stackrel{(b)}{\geq} \frac{4}{3} \min\left(\frac{1+l}{2+S\frac{1}{S}}, \frac{\frac{l^2}{1+l}}{2(1+S\delta_1)(1+S_{4,b,t})}\right) \\ &\stackrel{(c)}{\geq} \frac{4}{3} \min\left(\frac{1+l}{2+l}, \frac{\frac{l^2}{1+l}}{4(1+2l)}\right) \\ &\stackrel{(d)}{\geq} \frac{4}{3} \min\left(\frac{2}{3}, \frac{1}{24}\right) = \frac{1}{18}, \end{aligned}$$

where the bounds are obtained by: (a) using (161) and $\delta_2 \leq \frac{1}{1+l}$ and (147); (b) using $\delta_2 \leq \frac{1}{S}$ from (146) and $(1+S\delta_2+l\frac{1}{1+l}) \leq 2(1+S\delta_2)(1+S_{4,b,t})$; (c) using (158d) we have $(1+S\delta_2)(1+S_{4,b,t}) = (1+S_{4,a,t})(1+S_{4,b,t})$ and then using (148); and (d) come from minimizing over $l \geq 1$.

For receiver 2 we have

$$\frac{S(1-\delta_2)}{N_2^2-1} \geq \frac{S\frac{1}{1+l}}{N_2^2-1} \geq \frac{S\frac{1}{1+l}}{\frac{3}{4}\frac{S}{1+l}} = \frac{4}{3}l,$$

and therefore

$$\begin{aligned}
& \frac{d_{\min(S_2)}^2}{12} \\
&= \frac{1}{1 + S\delta_2 + l\delta_1} \min \left(\frac{(1 - \delta_2)S}{N_2^2 - 1}, \frac{(1 - \delta_1)l}{N_1^2 - 1} \right) \\
&\stackrel{(a)}{\geq} \frac{1}{1 + S\delta_2 + l\frac{1}{S}} \min \left(\frac{(1 - \frac{1}{1+l})S}{N_2^2 - 1}, \frac{(1 - \frac{1}{S})l}{N_1^2 - 1} \right) \\
&\stackrel{(b)}{\geq} \frac{4}{3} \frac{1}{1 + S\delta_2 + l\frac{1}{S}} \min \left(\frac{(1 - \frac{1}{1+l})S}{\frac{S}{1+l}}, \frac{(1 - \frac{1}{S})l(1+l)}{S-l} \right) \\
&= \frac{4}{3} \frac{1}{1 + S\delta_2 + l\frac{1}{S}} \min \left(l, \frac{l(1+l)}{S} \right) \\
&= \frac{4}{3} \min \left(\frac{l}{1 + S\delta_2 + l\frac{1}{S}}, \frac{l(1+l)}{(1 + S\delta_2 + l\frac{1}{S})S} \right) \\
&\stackrel{(c)}{\geq} \frac{4}{3} \min \left(\frac{l}{1 + S\frac{1}{1+l} + l\frac{1}{S}}, \frac{l(1+l)}{(1 + S\delta_2 + l\frac{1}{S})S} \right) \\
&\stackrel{(d)}{\geq} \frac{4}{3} \min \left(\frac{l}{1 + 2l}, \frac{l(1+l)}{(1 + \frac{(1+2l)(1+l)}{S} + l\frac{1}{S})S} \right) \\
&= \min \left(\frac{l}{1 + 2l}, \frac{l(1+l)}{(S + (1 + 2l)(1+l) + l^2)} \right) \\
&\stackrel{(e)}{\geq} \frac{4}{3} \min \left(\frac{l}{1 + 2l}, \frac{l(1+l)}{(l(1+l) + (1 + 2l)(1+l) + l^2)} \right) \\
&\geq \min \left(\frac{1}{3}, \frac{2}{10} \right) = \frac{4}{15},
\end{aligned}$$

where the bounds are obtained by: (a) using $\delta_2 \leq \frac{1}{1+l}$ and $\delta_1 \leq \frac{1}{S}$; (b) from (158a) we have that $N_1^2 - 1 \leq \frac{3}{4} \frac{S-l}{1+l}$; (c) using $\delta_2 \leq \frac{1}{1+l}$; (d) used bound in (160); and (e) used bound $S \leq l(1+l)$.

So, finally the gap is

$$\begin{aligned}
\text{gap}_{(162)} &\leq \max(\Delta_{R_1}, \Delta_{R_2}) \\
&= \frac{1}{2} \log \left(\frac{2^7}{3^2} \right) + \frac{1}{2} \log \left(\frac{\pi e}{3} \right) + \frac{1}{2} \log \left(1 + \frac{15}{4} \right) \\
&= \frac{1}{2} \log \left(\frac{608 \pi e}{27} \right) = \text{Gap} \left(\frac{608}{9}, 0 \right) \approx 3.79 \text{ bits.} \tag{162}
\end{aligned}$$

Overall Constant Gap for Weak 1: Therefore, the overall gap for Weak 1 is

$$\text{gap} \leq \max(\text{gap}_{(157)}, \text{gap}_{(162)}) = \text{gap}_{(157)}.$$

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